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Wavelet para-bases and sampling numbers in function spaces on domains

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Abstract

This paper deals with wavelet frames (para-bases), local polynomial reproducing formulas, and sampling numbers in function spaces on arbitrary and on E-thick domains in Euclidean n-space. In an Appendix we collect some recent instruments for corresponding function spaces on Euclidean n-space.

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1. Introduction

Unique wavelet representations in the function spaces $B^s_{pq}(\mathbb{R}^n)$ and $F^s_{pq}(\mathbb{R}^n)$ are known for all admitted parameters $s \in \mathbb{R}$, $0 < p \leqslant \infty$, ($p < \infty$ for the F-spaces), $0 < q \leqslant \infty$. They are unconditional bases if $p < \infty$, $q < \infty$. The situation for corresponding spaces on domains Ω in $\mathbb{R}^n$ is less favourable even if Ω is an interval or a cube and even if only classical function spaces are considered. But this problem attracted a lot of attention. The state of the art may be found in [1–4,8,10]. In [21,20, Section 4.2] we offered a new approach for some (sub-)spaces of $B^s_{pq}(\Omega)$ and $F^s_{pq}(\Omega)$ for bounded Lipschitz domains Ω in $\mathbb{R}^n$. This resulted in what we called para-bases. It is one aim of this paper to extend these considerations to more general (and more natural) domains Ω in $\mathbb{R}^n$. But we shift a comprehensive study of these problems to a later occasion restricting ourselves here to those assertions needed for the second (and main) purpose of this paper. We wish to demonstrate the symbiotic relationship between the recent theory of function spaces and some questions of numerical analysis such as local polynomial reproducing formulas and the

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accuracy of reconstructing functions belonging to some function spaces by the means of function values resulting in (linear and non-linear) sampling numbers. This is the direct continuation of and [12,20, Sections 4.3, 4.4].

For the reasons just outlined this is not a paper about (general) function spaces. This may justify to collect what we need in Appendix A. In addition to basic definitions we describe there those instruments of the recent theory of function spaces on $\mathbb{R}^n$ from which we believe that they complement more classical tools (such as derivatives and differences) in a decisive way. We hope that this Appendix may also serve as a little specific self-contained survey. We give references, but some assertions are formulated here for the first time, at least in the sharp versions presented.

The paper is organised as follows. Section 2 deals with refined localisation spaces $\overline{F}^s_{pq}(\Omega)$ on arbitrary domains Ω in $\mathbb{R}^n$ and characterisations in terms of wavelet para-bases. But first we remind of (classical and fractional) Sobolev spaces, classical Besov spaces and Hölder–Zygmund spaces as special cases of the spaces B^s_{pq} and F^s_{pq}. A reader who is not familiar with the theory of more general spaces may simply identify what follows with these special cases. In Section 3 we introduce E-thick domains (with bounded Lipschitz domains and snowflake domains as distinguished examples) and consider wavelet para-bases for the related spaces $\widetilde{B}^s_{pq}(\Omega)$ and $\widetilde{F}^s_{pq}(\Omega)$. Section 4 deals with wavelet J-para-bases in related B-spaces and F-spaces and respective local polynomial reproducing formulas. Clipping all together we arrive finally in Section 5 at sampling numbers of compact embeddings of some of these spaces into $L_t(\Omega)$ with $0 < t \leqslant \infty$.

2. Spaces on arbitrary domains

2.1. Distinguished spaces

We use the notation according to Appendix A, including Definition 24 where we introduced the spaces $B^s_{pq}(\mathbb{R}^n)$ and $F^s_{pq}(\mathbb{R}^n)$. But we describe a few distinguished special cases. A reader who is not familiar with the general spaces $B^s_{pq}(\mathbb{R}^n)$ and $F^s_{pq}(\mathbb{R}^n)$ may identify what follows with these special cases.

(i) Recall the Paley–Littlewood theorem

$$L_p(\mathbb{R}^n) = F^0_{p,2}(\mathbb{R}^n), \quad 1 < p < \infty. \tag{2.1}$$

(ii) Furthermore,

$$F^s_{p,2}(\mathbb{R}^n) = W^s_p(\mathbb{R}^n), \quad s \in \mathbb{N}_0, \ 1 < p < \infty, \tag{2.2}$$

are the *classical Sobolev spaces* usually normed by

$$\|f\,|W^s_p(\mathbb{R}^n)\| = \left(\sum_{|\alpha| \leqslant s} \|D^\alpha f\,|L_p(\mathbb{R}^n)\|^p \right)^{1/p}.$$

(iii) Recall that

$$I_\sigma : \quad f \mapsto \left(\langle \xi \rangle^\sigma \widehat{f} \right)^\vee, \quad \sigma \in \mathbb{R}, \ \langle \xi \rangle = (1 + |\xi|^2)^{1/2},$$

is a one-to-one map of $S(\mathbb{R}^n)$ onto itself and of $S'(\mathbb{R}^n)$ onto itself. It is a lift in the spaces $B^s_{pq}(\mathbb{R}^n)$ and $F^s_{pq}(\mathbb{R}^n)$,

$$I_\sigma B^s_{pq}(\mathbb{R}^n) = B^{s-\sigma}_{pq}(\mathbb{R}^n), \quad I_\sigma F^s_{pq}(\mathbb{R}^n) = F^{s-\sigma}_{pq}(\mathbb{R}^n),$$

for all admitted s, p, q. In particular,

$$H^s_p(\mathbb{R}^n) = I_{-s} L_p(\mathbb{R}^n), \quad s \in \mathbb{R}, \ 1 < p < \infty,$$

are the (fractional) *Sobolev spaces* with the classical Sobolev spaces

$$H^s_p(\mathbb{R}^n) = W^s_p(\mathbb{R}^n), \quad s \in \mathbb{N}_0, \ 1 < p < \infty,$$

as special cases.

(iv) Let

$$\left(\Delta^1_h f\right) = f(x+h) - f(x), \quad \left(\Delta^{l+1}_h f\right)(x) = \Delta^1_h \left(\Delta^l_h\right)(x), \tag{2.3}$$

where $x \in \mathbb{R}^n$, $h \in \mathbb{R}^n$, $l \in \mathbb{N}$, be iterated differences in $\mathbb{R}^n$. Then the *Hölder–Zygmund spaces* $\mathcal{C}^s(\mathbb{R}^n)$, $s > 0$, can be (equivalently) normed by

$$\| f \,|\mathcal{C}^s(\mathbb{R}^n)\|_m = \sup_{x \in \mathbb{R}^n} |f(x)| + \sup |h|^{-s} \, |\Delta^m_h f(x)|,$$

where $0 < s < m \in \mathbb{N}$. The second supremum is taken over all $x \in \mathbb{R}^n$ and all $h \in \mathbb{R}^n$ with $0 < |h| \leqslant 1$. One has

$$\mathcal{C}^s(\mathbb{R}^n) = B^s_{\infty\infty}(\mathbb{R}^n), \quad s > 0. \tag{2.4}$$

(v) The last assertion can be generalised as follows. Let

$$0 < p \leqslant \infty, \quad 0 < q \leqslant \infty, \quad \sigma_p < s < m \in \mathbb{N},$$

with σ_p as in (A.8). Then $B^s_{pq}(\mathbb{R}^n)$ can be equivalently quasi-normed by

$$\| f \,|B^s_{pq}(\mathbb{R}^n)\|_m = \| f \,|L_p(\mathbb{R}^n)\| + \left(\int_{|h| \leqslant 1} |h|^{-sq} \, \left\| \Delta^m_h f \,|L_p(\mathbb{R}^n)\right\|^q \, \frac{\mathrm{d}h}{|h|^n} \right)^{1/q}$$

(with the usual modification if $q = \infty$). If $1 \leqslant p < \infty$, $1 \leqslant q \leqslant \infty$, then $B^s_{pq}(\mathbb{R}^n)$ are the *classical Besov spaces*.

Remark 1. Similar lists and (historical) references may be found in [15, Section 2.2.2; 20, Section 1.2].

2.2. Refined localisation spaces

Open sets in $\mathbb{R}^n$ are denoted as domains. The refined localisation we have in mind is based on the well-known *Whitney decomposition*, applied to arbitrary domains Ω in $\mathbb{R}^n$ with $\Omega \neq \mathbb{R}^n$, in the version of Stein [13, Theorem 3, p. 16; Theorem 1, p. 167] adapted to our needs. Let

$$Q^0_{lr} \subset Q^1_{lr} \subset Q^2_{lr} \subset Q_{lr}, \quad l \in \mathbb{N}_0, \ r = 1, 2, \ldots, \tag{2.5}$$

be concentric open cubes in $\mathbb{R}^n$ with sides parallel to the axes of coordinates centred at $2^{-l} m^r$ for some $m^r \in \mathbb{Z}^n$ and with the respective side-lengths $2^{-l}, 5 \cdot 2^{-l-2}, 6 \cdot 2^{-l-2}, 2^{-l+1}$. According to the Whitney decomposition there are pairwise disjoint cubes Q^0_{lr} of this type such that

$$\Omega = \bigcup_{l,r} \overline{Q^0_{lr}} \quad \text{and} \quad \mathrm{dist}(Q_{lr}, \partial\Omega) \sim 2^{-l} \tag{2.6}$$

if $l \in \mathbb{N}$ and $r = 1, 2, \ldots$, complemented by $\mathrm{dist}(Q_{0r}, \partial\Omega) \geqslant c$ for some $c > 0$. We may assume $|l - l'| \leqslant 1$ for two adjacent cubes Q^0_{lr}, $Q^0_{l'r'}$. Let $\varrho = \{\varrho_{lr}\}$ be a related resolution of unity by non-negative C^∞ functions such that

$$\mathrm{supp}\, \varrho_{lr} \subset Q^1_{lr}, \quad |D^\gamma \varrho_{lr}(x)| \leqslant c_\gamma 2^{l|\gamma|}, \ \ \gamma \in \mathbb{N}^n_0, \tag{2.7}$$

for some $c_\gamma > 0$, and

$$\sum_{l=0}^{\infty} \sum_r \varrho_{lr}(x) = 1 \quad \text{if } \ x \in \Omega. \tag{2.8}$$

Let temporarily $F^s_{\infty\infty} = B^s_{\infty\infty}$. As usual, $D'(\Omega)$ is the collection of all distributions on Ω.

Definition 2. Let Ω be an arbitrary domain in $\mathbb{R}^n$ with $\Omega \neq \mathbb{R}^n$. Let $0 < p \leqslant \infty$, $0 < q \leqslant \infty$ (with $q = \infty$ if $p = \infty$) and $s > \sigma_{pq}$. Then

$$\overline{F}^s_{pq}(\Omega) = \left\{ f \in D'(\Omega) : \ \|f \,|\overline{F}^s_{pq}(\Omega)\|_\varrho < \infty \right\} \tag{2.9}$$

with

$$\|f \,|\overline{F}^s_{pq}(\Omega)\|_\varrho = \left(\sum_{l=0}^{\infty} \sum_r \|\varrho_{lr} f \,|F^s_{pq}(\mathbb{R}^n)\|^p \right)^{1/p} \tag{2.10}$$

(usual modification if $p = q = \infty$).

Remark 3. Of course $\varrho_{lr} f$ with $f \in D'(\Omega)$ is extended by zero outside of Ω. These spaces have a little history. In [17, Theorem 5.14] we proved for bounded C^∞ domains Ω in $\mathbb{R}^n$ that (2.10) is an equivalent quasi-norm in the closed subspaces

$$\left\{ f \in F^s_{pq}(\mathbb{R}^n) : \ \mathrm{supp}\, f \subset \overline{\Omega} \right\}$$

of $F^s_{pq}(\mathbb{R}^n)$, denoted as refined localisation property. We extended this assertion in [21,20, Propositon 4.20] to bounded Lipschitz domains in $\mathbb{R}^n$ under the additional restriction $p > 1$, $q > 1$. We return to this point below but without this restriction. There is no counterpart for B^s_{pq}-spaces if $p \neq q$. Now we take (2.9), (2.10) as a definition and call $\overline{F}^s_{pq}(\Omega)$ refined localisation spaces. One has to prove that $\overline{F}^s_{pq}(\Omega)$ is independent of $\varrho = \{\varrho_{lr}\}$. Furthermore we wish to characterise these spaces in terms of the ball means

$$d^M_{t,u} f(x) = \left(t^{-n} \int_{|h| \leqslant t} |(\Delta^M_h f)(x)|^u \,\mathrm{d}h \right)^{1/u}, \quad x \in \mathbb{R}^n, \ \ t > 0, \tag{2.11}$$

where $0 < u \leqslant \infty$ (usual modification if $u = \infty$) and where $(\Delta^M_h f)(x)$ are the differences according to (2.3). Let

$$\delta(x) = \min\left(1, \mathrm{dist}(x, \partial\Omega)\right), \quad x \in \Omega.$$

As usual $B(x, t)$ denotes a ball in $\mathbb{R}^n$ centred at $x \in \mathbb{R}^n$ and of radius $t > 0$. For $M \in \mathbb{N}$ let $\varkappa$ with $0 < \varkappa < 1$ and $c > 0$ be numbers such that

$$B(x, Mt) \subset \Omega, \quad \mathrm{dist}\left(B(x, Mt), \partial\Omega\right) \geqslant c\delta(x)$$

for all $x \in \Omega$ and $0 < t \leqslant \varkappa\delta(x)$. Let $L_p(\Omega)$ with $0 < p \leqslant \infty$ be the usual quasi-Banach space with respect to the Lebesgue measure, quasi-normed by

$$\| f \,|L_p(\Omega)\| = \left(\int_\Omega |f(x)|^p \,\mathrm{d}x \right)^{1/p}$$

with the obvious modification if $p = \infty$.

Theorem 4. *Let Ω be an arbitrary domain in $\mathbb{R}^n$ with $\Omega \neq \mathbb{R}^n$. Let*

$$0 < p \leqslant \infty, \quad 0 < q \leqslant \infty \;\; (\textit{with } q = \infty \textit{ if } p = \infty), \;\; s > \sigma_{pq}.$$

(i) *Then $\overline{F}^s_{pq}(\Omega)$ is a quasi-Banach space. It is independent of $\varrho = \{\varrho_{lr}\}$ (equivalent quasi-norms). Let*

$$\max(1, p) < w \leqslant \infty, \quad s - \frac{n}{p} > -\frac{n}{w} \tag{2.12}$$

(*interpreted as $w = \infty$ if $p = \infty$*). *Then*

$$\overline{F}^s_{pq}(\Omega) \hookrightarrow L_w(\Omega). \tag{2.13}$$

(ii) *Let $0 < u < \min(1, p, q)$ and $s < M \in \mathbb{N}$ in* (2.11). *Let $\varkappa$ be as above. Then $f \in L_w(\Omega)$* (*with w as in* (2.12)) *belongs to $\overline{F}^s_{pq}(\Omega)$ if, and only if,*

$$\left\| \left(\int_0^{\varkappa\delta(\cdot)} t^{-sq}\, d^M_{t,u} f(\cdot)^q \,\frac{\mathrm{d}t}{t} \right)^{1/q} \middle| L_p(\Omega) \right\| + \left\| \delta^{-s}(\cdot)\, f \middle| L_p(\Omega) \right\| < \infty \tag{2.14}$$

(*equivalent quasi-norms*).

Proof. The independence of $\overline{F}^s_{pq}(\Omega)$ of ϱ follows from the pointwise multiplier assertion in Proposition 42(ii). Furthermore, (2.13) follows from a corresponding assertion for $F^s_{pq}(\mathbb{R}^n)$, the obvious refined localisation property for $L_w(\Omega)$ and $p \leqslant w$. Finally, (2.14) is essentially covered by [17, Corollary 5.15, p. 66] and the underlying proof. □

Corollary 5. *Let Ω be an arbitrary domain in $\mathbb{R}^n$ with $\Omega \neq \mathbb{R}^n$. Then*

$$\overline{W}^k_p(\Omega) = \overline{F}^k_{p,2}(\Omega), \quad k \in \mathbb{N}, \;\; 1 < p < \infty,$$

is the collection of all $f \in L_p(\Omega)$ such that

$$\sum_{|\alpha| \leqslant k} \left\| \delta^{-k+|\alpha|} D^\alpha f \,\middle|\, L_p(\Omega) \right\| \sim \sum_{|\alpha| = k} \left\| D^\alpha f \,\middle|\, L_p(\Omega) \right\| + \left\| \delta^{-k} f \,\middle|\, L_p(\Omega) \right\| < \infty$$

(*equivalent norms*).

Proof. This follows from (2.2), (2.10) and well-known equivalent norms for the classical Sobolev spaces. One may also consult [14, Chapter 3]. □

2.3. Para-bases

It is one of the main aims of this paper to extend wavelet representations for B-spaces and F-spaces on $\mathbb{R}^n$ according to Theorem 40 to some F-spaces on arbitrary domains in Section 2 and to some B-spaces and F-spaces on E-thick domains in the next Section 3. To prepare both we introduce first wavelets and sequence spaces.

We always assume that Ω is an arbitrary domain in $\mathbb{R}^n$ with $\Omega \neq \mathbb{R}^n$ furnished with the Whitney decomposition (2.5), (2.6). We rely on the notation introduced in Section A.2.3 and modify (A.22) by

$$\Psi^j_{G,m}(x) = 2^{(j+L)n/2} \prod_{a=1}^{n} \psi_{G_a}\left(2^{j+L}x_a - m_a\right), \quad G \in \{F, M\}^n, \;\; m \in \mathbb{Z}^n, \tag{2.15}$$

where $L \in \mathbb{N}_0$ is fixed once and for all such that

$$\operatorname{supp} \Psi^j_{G,m} \subset Q_{lr} \quad \text{if } 2^{-j-L}m \in Q^2_{lr} \;\text{ for } l \in \mathbb{N}_0 \;\text{ and }\; j \geqslant l, \tag{2.16}$$

and

$$2^{-L-j}m \in Q^2_{lr} \quad \text{if } Q^1_{lr} \cap \operatorname{supp} \Psi^j_{G,m} \neq \emptyset \quad \text{for } l \in \mathbb{N}_0 \;\text{ and }\; j \geqslant l \tag{2.17}$$

for all admitted cubes according to (2.5), (2.6). We use the same notation as in (A.22) since one simply replaces the scaling function ψ_F in (A.18) by the scaling function $2^{L/2}\psi_F(2^L\cdot)$. With $\{F, M\}^n$ and $\{F, M\}^{n*}$ as in (A.20), (A.21) we introduce for $j \in \mathbb{N}_0$ the *main index set*

$$S^{\Omega,1}_j = \{F, M\}^{n*} \times \left\{ m \in \mathbb{Z}^n : \; 2^{-j-L}m \in Q^2_{lr} \text{ for some } l < j, \text{ some } r \right\} \tag{2.18}$$

and the *residual index set*

$$S^{\Omega,2}_j = \{F, M\}^n \times \left\{ m \in \mathbb{Z}^n : \; 2^{-j-L}m \in Q^2_{jr} \text{ for some } r \right\} \Big\backslash \, S^{\Omega,1}_j. \tag{2.19}$$

With

$$S^\Omega = S^{\Omega,1} \cup S^{\Omega,2}, \quad S^{\Omega,1} = \bigcup_{j=0}^{\infty} S^{\Omega,1}_j, \quad S^{\Omega,2} = \bigcup_{j=0}^{\infty} S^{\Omega,2}_j, \tag{2.20}$$

let

$$\Psi^{1,\Omega} = \left\{ \Psi^j_{G,m} : \; (j, G, m) \in S^{\Omega,1} \right\} \tag{2.21}$$

be the *main wavelet system* and

$$\Psi^{2,\Omega} = \left\{ \Psi^j_{G,m} : \; (j, G, m) \in S^{\Omega,2} \right\} \tag{2.22}$$

be the *residual wavelet system* where $\Psi^j_{G,m}$ are given by (2.15)–(2.17). This is an adapted version of corresponding constructions in [21,20, Section 4.2.4] where one finds further discussions, especially about the orthogonality of the systems $\Psi^{1,\Omega}$ and $\Psi^{2,\Omega}$. Let χ_{lr} be the characteristic functions of the cubes Q_{lr} in (2.5), (2.6).

Definition 6. Let Ω be an arbitrary domain in $\mathbb{R}^n$ with $\Omega \neq \mathbb{R}^n$. Let S^Ω be as in (2.18)–(2.20) and

$$S_j^\Omega = S_j^{\Omega,1} \cup S_j^{\Omega,2} \quad \text{with} \quad j \in \mathbb{N}_0.$$

Let $s \in \mathbb{R}$, $0 < p \leqslant \infty$, $0 < q \leqslant \infty$. Then $b_{pq}^{s,\Omega}$ is the collection of all sequences

$$\lambda = \left\{ \lambda_m^{j,G} \in \mathbb{C} : (j, G, m) \in S^\Omega \right\} \tag{2.23}$$

such that

$$\|\lambda \,|b_{pq}^{s,\Omega}\| = \left(\sum_{j=0}^{\infty} 2^{j(s-n/p)q} \left(\sum_{(G,m)\in S_j^\Omega} |\lambda_m^{j,G}|^p \right)^{q/p} \right)^{1/q} < \infty$$

and $f_{pq}^{s,\Omega}$ is the collection of all sequences (2.23) such that

$$\|\lambda \,| f_{pq}^{s,\Omega}\| = \left\| \left(\sum_{(j,G,m)\in S^\Omega} 2^{jsq} \left| \lambda_m^{j,G} \chi_{jm}(\cdot) \right|^q \right)^{1/q} \middle| L_p(\Omega) \right\| < \infty \tag{2.24}$$

(obviously modified if $p = \infty$ and/or $q = \infty$).

Remark 7. This is the Ω-version of Definition 38. We ask for the counterpart of Theorem 40, (A.24), (A.25), adapting [20, Section 4.2.4] where one finds some additional information. Let

$$\lambda_m^{j,G}(f) = 2^{jn/2} \int_\Omega f(x) \Psi_{G,m}^j(x)\, \mathrm{d}x, \quad (j, G, m) \in S^{\Omega,1}, \tag{2.25}$$

and

$$\lambda_m^{j,G}(f) = 2^{jn/2} \int_\Omega f(x) \varrho_m^j(x) \Psi_{G,m}^j(x)\, \mathrm{d}x, \quad (j, G, m) \in S^{\Omega,2}, \tag{2.26}$$

where ϱ_m^j are some non-negative C^∞ functions with

$$\operatorname{supp} \varrho_m^j \subset Q_{jr}, \quad |D^\gamma \varrho_m^j(x)| \leqslant c_\gamma 2^{j|\gamma|}, \quad \gamma \in \mathbb{N}_0^n, \tag{2.27}$$

where $r = r(m)$ has the same meaning as in (2.19).

Theorem 8. *Let Ω be an arbitrary domain in $\mathbb{R}^n$ with $\Omega \neq \mathbb{R}^n$. Let $0 < p \leqslant \infty$, $0 < q \leqslant \infty$ (with $q = \infty$ if $p = \infty$) and $\sigma_{pq} < s < u \in \mathbb{N}$. Let*

$$\Psi^\Omega = \Psi^{1,\Omega} \cup \Psi^{2,\Omega} = \left\{ \Psi_{G,m}^j : (j, G, m) \in S^\Omega \right\} \tag{2.28}$$

be the intrinsic wavelet system according to (2.18)–(2.22) *based on* ψ_F and ψ_M *according to* (A.18), (A.19). *Let $\overline{F}{}^s_{pq}(\Omega)$ be the spaces introduced in Definition* 2 *and let w be as in* (2.12), (2.13). *Then $f \in L_w(\Omega)$ is an element of $\overline{F}{}^s_{pq}(\Omega)$ if, and only if, it can be represented by*

$$f = \sum_{(j,G,m)\in S^\Omega} \lambda_m^{j,G} 2^{-jn/2} \Psi_{G,m}^j, \quad \lambda \in f_{pq}^{s,\Omega}. \tag{2.29}$$

Furthermore,

$$\|f\,|\overline{F}^s_{pq}(\Omega)\| \sim \inf \|\lambda\,|f^{s,\Omega}_{pq}\|, \tag{2.30}$$

where the infimum is taken over all representations (2.29). *Any* $f \in \overline{F}^s_{pq}(\Omega)$ *can be represented by*

$$f = \sum_{(j,G,m)\in S^\Omega} \lambda^{j,G}_m(f)\,2^{-jn/2}\,\Psi^j_{G,m} \tag{2.31}$$

with (2.25)–(2.27), *where* ϱ^j_m *is the sum of some functions of the system* $\varrho = \{\varrho_{lr}\}$ *and*

$$\|f\,|\overline{F}^s_{pq}(\Omega)\| \sim \|\lambda(f)\,|f^{s,\Omega}_{pq}\| \tag{2.32}$$

(*equivalent quasi-norms*).

Proof. *Step* 1: Since $\sigma_{pq} < s < u \in \mathbb{N}$ one gets by Theorem 30 that (2.29), after correct normalisation, is an atomic decomposition in $F^s_{pq}(\mathbb{R}^n)$. No moment conditions are needed for the atoms $2^{-jn/2}2^{-j(s-n/p)}\Psi^j_{G,m}$. This applies also to

$$\varrho_{lr} f = \sum_{(j,G,m)\in S^\Omega} \lambda^{j,G}_m\,2^{-jn/2}\varrho_{lr}\,\Psi^j_{G,m}$$

with ϱ_{lr} as in (2.7), (2.8). Then it follows from Theorem 30, (2.10) and the structure of $f^{s,\Omega}_{pq}$ that

$$\|f\,|\overline{F}^s_{pq}(\Omega)\|^p \sim \sum_{l,r} \|\varrho_{lr} f\,|F^s_{pq}(\mathbb{R}^n)\|^p \leqslant c\|\lambda\,|f^{s,\Omega}_{pq}\|^p \tag{2.33}$$

(modification if $p = q = \infty$).

Step 2: As for the converse we apply first the homogeneity property from Proposition 42 to $\varrho_{lr} f$ with $\varepsilon = 2^{-l}$. Then we expand each $(\varrho_{lr} f)(2^{-l}\cdot)$ according to Theorem 40(ii) where now $u > s$ is sufficient. Clipping together the re-transformed expansions one gets

$$f = \sum_{l,r} \varrho_{lr} f = \sum_{(j,G,m)\in S^\Omega} \lambda^{j,G}_m(f)\,2^{-jn/2}\Psi^j_{G,m}$$

with (2.25)–(2.27), hence (2.31) and (2.32). As for some details about these scaling procedures we refer to [20, Section 4.2.2, especially Proposition 4.17]. Then one gets by (2.33) also (2.30). □

Remark 9. If $(j, G, m) \in S^{\Omega,1}$ then the coefficients $\lambda^{j,G}_m$ in (2.29) are unique and they coincide with $\lambda^{j,G}_m(f)$ in (2.25). The summation over $S^{\Omega,1}$ in (2.29) remains to be (after appropriate normalisation) an expansion by an orthonormal basis. The coefficients in the summation over the residual part $S^{\Omega,2}$ might not be unique, but this part is harmless by its construction. In any case (2.31) with (2.25)–(2.27) is a stable frame (where stable refers to the optimality of $\lambda(f)$ according to (2.30)). This may justify to call Ψ^Ω in (2.28) a para-basis. Further details may be found in [20, Section 4.2.4] including discussions about the convergence of (2.29). We collect the outcome which can also be obtained directly from (2.29). One has always unconditional convergence in $S'(\mathbb{R}^n)$. If $p < \infty$ and $w < \infty$ in (2.12) then (2.29) converges absolutely (and hence unconditionally) in $L_w(\Omega)$. If $p < \infty$, $q < \infty$ then (2.29) converges unconditionally in

$\overline{F}^s_{pq}(\Omega)$. If $p<\infty, q=\infty$ then one has unconditional convergence in $\overline{F}^\sigma_{pp}(\Omega)$ with $\sigma_{pq}<\sigma<s$. If Ω is bounded and $p=q=\infty$, then (2.29) converges unconditionally in $\overline{\mathcal{C}}^\sigma(\Omega)=\overline{B}^\sigma_{\infty\infty}(\Omega)$ with $0<\sigma<s$ (using the notation (2.4)). If Ω is unbounded then one has this convergence at least in any domain $\{x\in\Omega, |x|<R\}$ with $0<R\ (\to\infty)$. For our later considerations we need a counterpart of Theorem 8 for $L_r(\Omega)$ with $1<r<\infty$.

Theorem 10. *Let Ω be an arbitrary domain in $\mathbb{R}^n$ with $\Omega\neq\mathbb{R}^n$. Let $1<r<\infty$ and let*

$$\Psi^\Omega=\left\{\Psi^j_{G,m}:\ (j,G,m)\in S^\Omega\right\} \tag{2.34}$$

be the same intrinsic wavelet system as in (2.28) *based on ψ_F and ψ_M according to* (A.18), (A.19) *now with $u\in\mathbb{N}$. Then $L_r(\Omega)$ is the collection of all locally integrable functions f (in Ω) which can be represented by*

$$f=\sum_{(j,G,m)\in S^\Omega}\lambda^{j,G}_m\,2^{-jn/2}\,\Psi^j_{G,m},\quad \lambda\in f^{0,\Omega}_{r,2}. \tag{2.35}$$

Furthermore,

$$\|f\,|L_r(\Omega)\|\sim\inf\|\lambda\,|f^{0,\Omega}_{r,2}\|, \tag{2.36}$$

where the infimum is taken over all representations (2.35). *Any $f\in L_r(\Omega)$ can be represented by* (2.31) *with* (2.25)–(2.27) *and*

$$\|f\,|L_r(\Omega)\|\sim\|\lambda(f)\,|f^{0,\Omega}_{r,2}\| \tag{2.37}$$

(*equivalent norms*).

Proof. Obviously, $L_r(\Omega)=\overline{L}_r(\Omega)$ has the refined localisation property according to Definition 2 with L_r in place of F^s_{pq}. Furthermore, there is also an immediate counterpart of the homogeneity property (A.31). Using the Paley–Littlewood assertion (2.1) one can carry over Step 2 of the proof of Theorem 8 resulting in the representation (2.31) with (2.37). It remains to prove (2.36) for any representation (2.35). We split Ψ^Ω as in (2.28) in its main wavelet system and its residual wavelet system, hence

$$f=\sum_{(j,G,m)\in S^{\Omega,1}}\cdots+\sum_{(j,G,m)\in S^{\Omega,2}}\cdots=f_1+f_2.$$

According to Theorem 30 one needs first moment conditions for atoms in $L_r(\mathbb{R}^n)=F^0_{r,2}(\mathbb{R}^n)$. By (A.19) with $v=0$ it follows that $\Psi^j_{G,m}$ with $\Psi^j_{G,m}\in S^{\Omega,1}$ are atoms (after normalisation) with respect to $L_r(\mathbb{R}^n)$. We split λ in (2.35) into λ^1 and λ^2,

$$\lambda^l=\left\{\lambda^{j,G}_m\in\mathbb{C}:\ (j,G,m)\in S^{\Omega,l}\right\},\quad l=1,2.$$

Then it follows from Theorem 30 that

$$\|f_1\,|L_r(\Omega)\|\sim\|f_1\,|F^0_{r,2}(\mathbb{R}^n)\|\leqslant c\,\|\lambda^1\,|f^{0,\Omega}_{r,2}\|. \tag{2.38}$$

One has for the residual part f_2,

$$\begin{aligned} \| f_2 \,| L_r(\Omega)\|^r &\leqslant c \sum_{(j,G,m)\in S^{\Omega,2}} 2^{-jn} \, |\lambda_m^{j,G}|^r \\ &\sim \|\lambda^2 \,| f_{rr}^{0,\Omega}\|^r \sim \|\lambda^2 \,| f_{r,2}^{0,\Omega}\|^r \,, \end{aligned} \tag{2.39}$$

where we used the structure of $f_{rq}^{0,\Omega}$ according to (2.24) and the structure of $S^{\Omega,2}$. By (2.38) and (2.39) one gets the desired estimate

$$\| f \,| L_r(\Omega)\| \leqslant c \, \|\lambda \,| f_{r,2}^{0,\Omega}\|.$$

Together with (2.37) one gets (2.36). $\square$

Although not subject of this paper we mention a somewhat curious consequence of the last theorem. Let $k \in \mathbb{N}_0$ and $k \leqslant u \in \mathbb{N}$ where u has the same meaning as above. Put $\varrho_m^j = 1$ if $(j, G, m) \in S^{\Omega,1}$. Then we modify the absolute values of (2.25), (2.26) by

$$\lambda_m^{j,G}(f)^k = 2^{jn/2} \sum_{|\alpha|\leqslant k} \left| \int_\Omega f(x) \cdot D^\alpha \left(\varrho_m^j \Psi_{G,m}^j \right)(x) \, \mathrm{d}x \right|.$$

For $1 < r < \infty$ and $k \in \mathbb{N}_0$ let

$$W_r^k(\Omega) = \left\{ f \in L_r(\Omega) : \ D^\alpha f \in L_r(\Omega), \, |\alpha| \leqslant k \right\}$$

be the obviously normed intrinsically defined Sobolev spaces.

Corollary 11. *Let Ω be an arbitrary domain in $\mathbb{R}^n$ with $\Omega \neq \mathbb{R}^n$. Let $1 < r < \infty$, $k \in \mathbb{N}$, and let Ψ^Ω be as in* (2.34) *with $k \leqslant u$. Then $f \in L_r(\Omega)$ is an element of $W_r^k(\Omega)$ if, and only if, it can be represented by*

$$f = \sum_{(j,G,m)\in S^\Omega} \lambda_m^{j,G}(f)^k \, 2^{-jn/2} \, \Psi_{G,m}^j, \quad \lambda(f)^k \in f_{r,2}^{0,\Omega}$$

with (2.25)–(2.27). *Furthermore*,

$$\| f \,| W_r^k(\Omega)\| \sim \|\lambda(f)^k \,| f_{r,2}^{0,\Omega}\|$$

(*equivalent norms*).

Proof. This follows from Theorem 10 applied to $D^\alpha f$ with $|\alpha| \leqslant k$ and partial integration as far as the coefficients are concerned. $\square$

3. Spaces on E-thick domains

3.1. E-thick domains

Recall that domain means open set. Let $l(Q)$ be the side-length of a cube Q in $\mathbb{R}^n$ with sides parallel to the axes of coordinates.

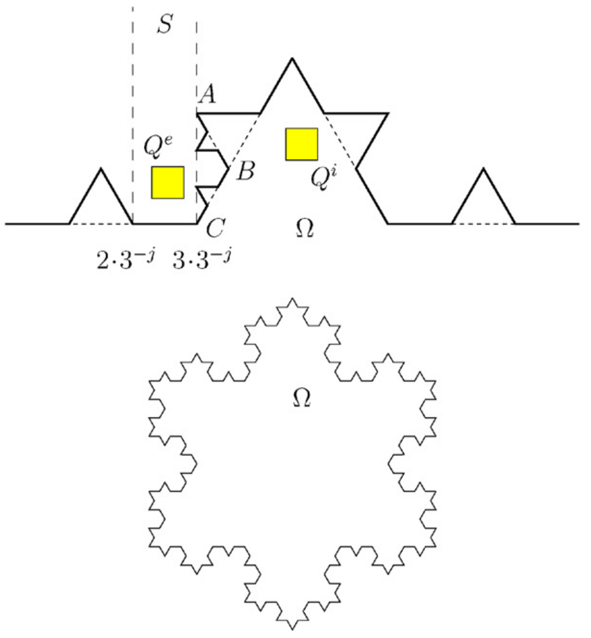

Fig. 1.

Definition 12. A domain Ω in $\mathbb{R}^n$ is said to be E-thick (exterior thick) if one finds for any interior cube $Q^i \subset \Omega$ with

$$l(Q^i) \sim 2^{-j}, \quad \mathrm{dist}(Q^i, \partial\Omega) \sim 2^{-j}, \quad j \geqslant j_0 \in \mathbb{N}, \tag{3.1}$$

a complementing exterior cube $Q^e \subset \Omega^c = \mathbb{R}^n \backslash \Omega$ with

$$l(Q^e) \sim 2^{-j}, \quad \mathrm{dist}(Q^e, \partial\Omega) \sim \mathrm{dist}(Q^i, Q^e) \sim 2^{-j},$$

where all equivalence constants are independent of j.

Example 13. Every bounded Lipschitz domain in $\mathbb{R}^n$ is E-thick. If Ω in $\mathbb{R}^2$ is (locally) above the cusp $x_2 = |x_1|^\alpha$ with $0 < \alpha < 1$ then Ω is (locally) E-thick. If Ω is (locally) below this cusp then Ω is not E-thick. As indicated in Fig. 1 the usual snowflake domain in $\mathbb{R}^2$ is E-thick. But there might be rather bizarre E-thick domains.

Proposition 14. (i) *For any domain Ω in $\mathbb{R}^n$ with $\Omega \neq \mathbb{R}^n$ one has*

$$\mathbb{R}^n = \Omega \cup \partial\Omega \cup (\mathbb{R}^n \backslash \overline{\Omega}) \quad \textit{and} \quad \partial\left(\mathbb{R}^n \big\backslash \overline{\Omega}\right) \subset \partial\Omega.$$

Furthermore,

$$\partial\Omega = \partial\left(\mathbb{R}^n \big\backslash \overline{\Omega}\right) \quad \textit{if, and only if}, \quad (\overline{\Omega})^\circ = \Omega.$$

(ii) *If Ω* is *E-thick then $(\overline{\Omega})^\circ = \Omega$.*

(iii) *There are bounded E-thick domains Ω with $|\partial\Omega| > 0$.*

Proof. One checks (i) and (ii) easily. We prove (iii). Let $\{r_l : l \in \mathbb{N}\}$ be the set of all rational numbers with $0 < r_l < 1$ and let I_l be open intervals centred at r_l such that $I_l \subset (0,1)$. Let

$$\Omega^0 = \bigcup_{l=1}^{\infty} I_l = \bigcup_{l=1}^{\infty} I_l^0 \quad \text{with} \sum_{l=1}^{\infty} |I_l| < 1,$$

where I_l^0 are disjoint open intervals. Then

$$\partial\Omega^0 = [0,1] \Big\backslash \bigcup_{l=1}^{\infty} I_l \quad \text{and} \quad |\partial\Omega^0| > 0.$$

We decompose each interval I_l^0 into

$$I_l^0 = I_l^1 \cup \left\{ x_l^k \right\}_{k=1}^{\infty} \cup I_l^2,$$

where I_l^1 is the union of disjoint open intervals $I_{l,k}^1$ of length, say, $\sim 2^{-k}|I_l^0|$, $k \in \mathbb{N}$. Similarly I_l^2. This can be done in such a way that I_l^1 is E-thick at the expense of I_l^2 and vice versa. Then $\Omega^1 = \bigcup I_l^1$ is E-thick at the expense of $\Omega^2 = \bigcup I_l^2$ and vice versa. Furthermore,

$$0 < |\partial\Omega^0| = |\partial\Omega^1| + |\partial\Omega^2|.$$

This proves (iii). □

3.2. Spaces and para-bases

Recall again that domain means open set. As usual, $D'(\Omega)$ is the collection of all distributions on the domain Ω.

Definition 15. Let Ω be an arbitrary domain in $\mathbb{R}^n$ with $\Omega \neq \mathbb{R}^n$. Let $s \in \mathbb{R}$, $0 < p \leqslant \infty$ ($p < \infty$ for F-spaces), $0 < q \leqslant \infty$.

(i) Then $B^s_{pq}(\Omega)$ is the collection of all $f \in D'(\Omega)$ such that there is an $g \in B^s_{pq}(\mathbb{R}^n)$ with $g|\Omega = f$. Furthermore,

$$\| f \,| B^s_{pq}(\Omega) \| = \inf \| g \,| B^s_{pq}(\mathbb{R}^n) \|,$$

where the infimum is taken over all $g \in B^s_{pq}(\mathbb{R}^n)$ with $g|\Omega = f$. Similarly for $F^s_{pq}(\Omega)$.

(ii) Let

$$\widetilde{B}^s_{pq}(\overline{\Omega}) = \left\{ f \in B^s_{pq}(\mathbb{R}^n) : \operatorname{supp} f \subset \overline{\Omega} \right\}$$

and

$$\widetilde{F}^s_{pq}(\overline{\Omega}) = \left\{ f \in F^s_{pq}(\mathbb{R}^n) : \operatorname{supp} f \subset \overline{\Omega} \right\}$$

as closed subspaces of the corresponding spaces on $\mathbb{R}^n$.

(iii) Then $\widetilde{B}^s_{pq}(\Omega)$ is the collection of all $f \in D'(\Omega)$ such that there is an $g \in \widetilde{B}^s_{pq}(\overline{\Omega})$ with $g|\Omega = f$. Furthermore,

$$\| f \,| \widetilde{B}^s_{pq}(\Omega) \| = \inf \| g \,| \widetilde{B}^s_{pq}(\overline{\Omega}) \|,$$

where the infimum is taken over all $g \in \widetilde{B}^s_{pq}(\overline{\Omega})$ with $g|\Omega = f$. Similarly for $\widetilde{F}^s_{pq}(\Omega)$.

Remark 16. If $|\partial\Omega| = 0$ and $0 < p \leqslant \infty$, $0 < q \leqslant \infty$, $s > \sigma_p$, then one may identify the spaces in (ii) and (iii) (appropriately interpreted). On the other hand, according to Proposition 14(iii) there are bounded E-thick domains Ω with $|\partial\Omega| > 0$. Then even for $s > \sigma_p$ the spaces in (ii) and (iii) might be different. Next we deal with the counterpart of Theorem 8 using the same notation as there.

Theorem 17. *Let Ω be an E-thick domain in $\mathbb{R}^n$ according to Definition* 12.

(i) *Let $0 < p \leqslant \infty$, $0 < q \leqslant \infty$, $s > \sigma_p$, and let $\widetilde{B}^s_{pq}(\Omega)$ be as in Definition* 15(iii). *Let Ψ^Ω be the same intrinsic wavelet system as in* (2.28) *based on ψ_F and ψ_m according to* (A.18), (A.19) *with $s < u \in \mathbb{N}$. Let w be as in* (2.12). *Then*

$$\widetilde{B}^s_{pq}(\Omega) \hookrightarrow L_w(\Omega). \tag{3.2}$$

Let $b^{s,\Omega}_{pq}$ be as in Definition 6. *Then $f \in L_w(\Omega)$ is an element of $\widetilde{B}^s_{pq}(\Omega)$ if, and only if, it can be represented by*

$$f = \sum_{(j,G,m)\in S^\Omega} \lambda^{j,G}_m 2^{-jn/2} \Psi^j_{G,m}, \quad \lambda \in b^{s,\Omega}_{pq}. \tag{3.3}$$

Furthermore,

$$\|f\,|\widetilde{B}^s_{pq}(\Omega)\| \sim \inf \|\lambda\,|b^{s,\Omega}_{pq}\|,$$

where the infimum is taken over all representations (3.3). *Any $f \in \widetilde{B}^s_{pq}(\Omega)$ can be represented by*

$$f = \sum_{(j,G,m)\in S^\Omega} \lambda^{j,G}_m(f)\, 2^{-jn/2} \Psi^j_{G,m} \tag{3.4}$$

with (2.25)–(2.27) *and*

$$\|f\,|\widetilde{B}^s_{pq}(\Omega)\| \sim \|\lambda(f)\,|b^{s,\Omega}_{pq}\|$$

(*equivalent quasi-norms*).

(ii) *Let $0 < p < \infty$, $0 < q \leqslant \infty$, $s > \sigma_{pq}$, and let $\widetilde{F}^s_{pq}(\Omega)$ be as in Definition* 15(iii). *Let Ψ^Ω and w be as in part* (i) *again with $s < u \in \mathbb{N}$. Then one has* (3.2) *with F in place of B. Let $f^{s,\Omega}_{pq}$ be as in Definition* 6. *Then $f \in L_w(\Omega)$ is an element of $\widetilde{F}^s_{pq}(\Omega)$ if, and only if, it can be represented by*

$$f = \sum_{(j,G,m)\in S^\Omega} \lambda^{j,G}_m 2^{-jn/2} \Psi^j_{G,m}, \quad \lambda \in f^{s,\Omega}_{pq}. \tag{3.5}$$

Furthermore,

$$\|f\,|\widetilde{F}^s_{pq}(\Omega)\| \sim \inf \|\lambda\,|f^{s,\Omega}_{pq}\|,$$

where the infimum is taken over all representations (3.5). *Any $f \in \widetilde{F}^s_{pq}(\Omega)$ can be represented by* (3.4) *and*

$$\|f\,|\widetilde{F}^s_{pq}(\Omega)\| \sim \|\lambda(f)\,|f^{s,\Omega}_{pq}\|$$

(*equivalent quasi-norms*).

Proof. The embedding (3.2) and its F-counterpart follow from a corresponding assertion in $\mathbb{R}^n$. We prove (ii). First, we remark that (3.5) can be considered as an atomic decomposition according to Theorem 30(ii) (after correct normalisation), no moment conditions are needed. One gets

$$\|f\,|\widetilde{F}^s_{pq}(\Omega)\| \leqslant \|f\,|\widetilde{F}^s_{pq}(\overline{\Omega})\| = \|f\,|F^s_{pq}(\mathbb{R}^n)\| \leqslant c\,\|\lambda\,|f^{s,\Omega}_{pq}\|. \tag{3.6}$$

Conversely, let $f \in \widetilde{F}^s_{pq}(\Omega)$. Then $f \in L_w(\Omega)$ with $w < \infty$, and it follows from Theorem 10 that f can be represented by (3.4) with (2.25)–(2.27) at least in $L_w(\Omega)$. We wish to apply Theorem 36(ii) identifying (A.15) with (2.25), (2.26). By (A.18)–(A.22) one has the required moment conditions in (A.13) with $B = u > s$ if $G \in \{F, M\}^{n*}$. This applies to all kernels $\Psi^j_{G,m}$ in (2.25) according to (2.18), (2.20). The kernels $\varrho^j_m\,\Psi^j_{G,m}$ in (2.26) may not have the required moment conditions. Of course one has only to care for terms with $j \geqslant j_0$. In particular the kernels $\varrho^j_m\,\Psi^j_{G,m}$ in question have supports in cubes to which Definition 12 applies, hence

$$\operatorname{supp} \varrho^j_m\,\Psi^j_{G,m} \subset Q^i$$

with (3.1). Let Q^e be a related complementing exterior cube. Then there is a function $\widetilde{\Psi}^j_{G,m} \in C^u(\mathbb{R}^n)$ with $\operatorname{supp} \widetilde{\Psi}^j_{G,m} \subset Q^e$ such that

$$k^G_{jm}(x) = \varrho^j_m(x)\,\Psi^j_{G,m}(x) + \widetilde{\Psi}^j_{G,m}(x), \quad x \in \mathbb{R}^n,$$

is an admitted kernel satisfying the required moment conditions. The existence of such a complementing function $\widetilde{\Psi}^j_{G,m}$ is quite plausible but not obvious. We refer for details to [22, p. 665]. Let $g \in \widetilde{F}^s_{pq}(\overline{\Omega})$ with $g|\Omega = f$ and

$$\|g\,|F^s_{pq}(\mathbb{R}^n)\| = \|g\,|\widetilde{F}^s_{pq}(\overline{\Omega})\| \sim \|f\,|\widetilde{F}^s_{pq}(\Omega)\|.$$

Since $\operatorname{supp} g \subset \overline{\Omega}$ one gets

$$\int_{\mathbb{R}^n} k^G_{jm}(x)\,g(x)\,\mathrm{d}x = \int_{\mathbb{R}^n} \varrho^j_m(x)\,\Psi^j_{G,m}(x)\,f(x)\,\mathrm{d}x.$$

Now one can apply Theorem 36(ii) and obtains that

$$\|\lambda(f)\,|f^{s,\Omega}_{pq}\| \leqslant c\,\|g\,|F^s_{pq}(\mathbb{R}^n)\| \sim \|f\,|\widetilde{F}^s_{pq}(\Omega)\|. \tag{3.7}$$

Then (3.4), (3.7) and (3.6) prove part (ii). The proof of part (i) is the same. We only mention that we have now $w = \infty$ if $p = \infty$. But everything in representation (3.4) is local and applies also to $L_\infty(\Omega)$ since $L_\infty(\Omega) \subset L^{\mathrm{loc}}_v(\Omega)$ for $1 < v < \infty$. □

Corollary 18. *Let Ω be an E-thick domain in $\mathbb{R}^n$ according to Definition* 12 *and let*

$$0 < p \leqslant \infty, \quad 0 < q \leqslant \infty \ \ (\text{with } q = \infty \text{ if } p = \infty), \quad s > \sigma_{pq}.$$

Let $\overline{F}^s_{pq}(\Omega)$ be as in Definition 2 *and $\widetilde{F}^s_{pq}(\Omega)$ be as in Definition* 15(iii) (*with $\widetilde{F}^s_{\infty\infty} = \widetilde{B}^s_{\infty\infty}$*). *Then*

$$\overline{F}^s_{pq}(\Omega) = \widetilde{F}^s_{pq}(\Omega).$$

Proof. This is an immediate consequence of the Theorems 8 and 17.

4. J-para-bases and polynomial reproducing formulas

4.1. J-para-bases

We modified (A.22) in (2.15) by an additional dilation 2^L. This was not indicated since L is assumed to be fixed once and for all such that one has (2.16), (2.17) based on the Whitney decomposition (2.5), (2.6). Now we replace $l \geqslant 0$ in (2.5)–(2.8) and also in (2.16), (2.17) by $l \geqslant J \in \mathbb{N}_0$. Then we have to adapt the wavelet system (2.18)–(2.22) appropriately where we indicate now J. This is covered by the multi-resolution philosophy. We fix the outcome. Instead of (2.5), (2.6) we have now

$$Q^0_{lr} \subset Q^1_{lr} \subset Q^2_{lr} \subset Q_{lr}, \quad l \geqslant J, \quad r = 1, 2, \dots,$$

and

$$\Omega = \bigcup_{l,r} \overline{Q^0_{lr}} \quad \text{and} \quad \operatorname{dist}(Q_{lr}, \partial\Omega) \sim 2^{-l} \ \text{ if } l > J$$

and $r = 1, 2, \dots$, complemented by $\operatorname{dist}(Q_{Jr}, \partial\Omega) \geqslant c\, 2^{-J}$ for some $c > 0$. In (2.8) the summation over $l \in \mathbb{N}_0$ is now replaced by $l \geqslant J$. In (2.16), (2.17) we assume now $l \geqslant J$ with respect to (2.15) which remains unchanged. Similarly one has now (2.18) with $J \leqslant l < j$ and (2.19) with $J \leqslant j$ and as a consequence

$$(JS)^\Omega = (JS)^{\Omega,1} \cup (JS)^{\Omega,2}, \quad (JS)^{\Omega,1} = \bigcup_{j=J}^{\infty} S^{\Omega,1}_j, \quad (JS)^{\Omega,2} = \bigcup_{j=J}^{\infty} S^{\Omega,2}_j. \tag{4.1}$$

Then one gets an obvious modification of Theorem 8. In particular, $f \in \overline{F}^s_{pq}(\Omega)$ can be optimally represented as

$$f = \sum_{(j,G,m)\in (JS)^\Omega} \lambda^{j,G}_m(f)\, 2^{-jn/2}\, \Psi^j_{G,m} \tag{4.2}$$

with

$$\lambda^{j,G}_m(f) = 2^{jn/2} \int_\Omega f(x)\, \Psi^j_{G,m}(x)\, \mathrm{d}x \tag{4.3}$$

or

$$\lambda^{j,G}_m(f) = 2^{jn/2} \int_\Omega f(x)\, \varrho^j_m(x)\, \Psi^j_{G,m}(x)\, \mathrm{d}x \tag{4.4}$$

with ϱ^j_m as in (2.27). However, instead of the decomposition (4.1) we rely now on the decomposition of $(JS)^\Omega$ into three disjoint index sets,

$$(JS)^\Omega = \langle JS\rangle^\Omega \cup \{JS\}^\Omega \cup [JS]^\Omega,$$

where

- $\langle JS\rangle^\Omega$ collects all $(j, G, m) \in (JS)^\Omega$ where $j = J$ and $G = (F)^n = (F, \dots, F)$ with $\lambda^{j,G}_m(f)$ as in (4.3),
- $\{JS\}^\Omega$ collects all $(j, G, m) \in (JS)^\Omega$ where $j \geqslant J$ and $G \in \{F, M\}^{n*}$ with $\lambda^{j,G}_m(f)$ as in (4.3),

- $[JS]^\Omega$ collects the remaining elements $(j, G, m) \in (JS)^\Omega$.

  In particular, $\langle JS\rangle^\Omega$ refers to those n-dimensional father wavelets

$$\Psi^J_m(x) = \Psi^J_{(F)^n,m}(x) = 2^{(J+L)n/2} \prod_{a=1}^{n} \psi_F\left(2^{J+L}x_a - m_a\right),$$

where

$$\lambda^J_m(f) = \lambda^{J,(F)^n}_m(f) = 2^{Jn/2} \int_\Omega f(x)\, \Psi^J_m(x)\, \mathrm{d}x.$$

By construction this applies to all terms Ψ^J_m with

$$\mathrm{dist}(\mathrm{supp}\, \Psi^J_m, \partial\Omega) \geqslant c_1\, 2^{-J}, \quad J \in \mathbb{N}_0,$$

for some $c_1 > 0$ which is independent of J. Recall that ϱ^j_m in (4.4) is independent of admitted $G \in \{F, M\}^n$ and that

$$\mathrm{dist}(\mathrm{supp}\, \Psi^j_{G,m}, \partial\Omega) \leqslant c_2\, 2^{-J} \quad \text{if } (j, G, m) \in [JS]^\Omega, \tag{4.5}$$

for some $c_2 > 0$ which is independent of $J \in \mathbb{N}_0$. Hence (4.2) is now decomposed into three sums,

$$\begin{aligned} f &= \sum_{(j,G,m)\in(JS)^\Omega} \lambda^{j,G}_m(f)\, 2^{-jn/2}\, \Psi^j_{G,m} \\ &= \sum_{(j,G,m)\in\langle JS\rangle^\Omega} \lambda^J_m(f)\, 2^{-Jn/2}\, \Psi^J_m + \sum_{(j,G,m)\in\{JS\}^\Omega} \cdots + \sum_{(j,G,m)\in[JS]^\Omega} \cdots \\ &= \langle f\rangle_J + \{f\}_J + [f]_J, \end{aligned} \tag{4.6}$$

indicating $J \in \mathbb{N}_0$.

4.2. Local polynomial reproducing formulas

Decompositions of type (4.6) are the basis for local reproducing formulas. Let for $\tau > 0$,

$$\Omega_\tau = \left\{x \in \Omega :\ \mathrm{dist}(x, \partial\Omega) > \tau\right\} \tag{4.7}$$

and let $\overline{C}(\Omega)$ be the set of all (complex-valued) continuous bounded functions in the (arbitrary) domain Ω in $\mathbb{R}^n$. As usual, $B(x, \tau)$ stands for a ball in $\mathbb{R}^n$ centred at $x \in \mathbb{R}^n$ and of radius $\tau > 0$. Let $\mathcal{P}^M(\Omega)$ be the collection of all complex-valued polynomials of degree less than $M \in \mathbb{N}$ in Ω.

Theorem 19. *Let Ω be an arbitrary domain in Ω with $|\Omega| < \infty$. Let $\overline{F}{}^s_{pq}(\Omega)$ be the spaces according to Definition 2 with $0 < p \leqslant \infty$, $0 < q \leqslant \infty$ (with $q = \infty$ if $p = \infty$) and*

$$s > \max\left(\frac{n}{p},\ \sigma_{pq}\right).$$

Let $M \in \mathbb{N}$. Then there are numbers $\tau_0 > 0$, $a > 0$, $b > 0$, $c > 0$, with the following property. For any τ with $0 < \tau \leqslant \tau_0$ one finds points $x^j \in \Omega_\tau$, having pairwise distance of at least $a\tau$, and real functions $h^\tau_j \in \overline{C}(\Omega)$ with

$$\sup |h^\tau_j(x)| \leqslant c, \quad \mathrm{supp}\, h^\tau_j \subset B(x^j, b\tau) \subset \Omega_\tau, \tag{4.8}$$

such that the mapping U_τ,

$$U_\tau f = \sum_j f(x^j)\, h_j^\tau, \quad f \in \overline{F}{}^s_{pq}(\Omega), \tag{4.9}$$

is polynomial reproducing in Ω_τ,

$$(U_\tau P)(x) = P(x) \quad \text{where } P \in \mathcal{P}^M(\Omega), \;\; x \in \Omega_\tau. \tag{4.10}$$

Proof. *Step* 1: Let $u > \max(M-1, s)$ in (A.18), (A.19). Then we can apply Theorem 8 now based on (4.6). Furthermore, one has for $P \in \mathcal{P}^M(\Omega)$,

$$\lambda_m^{j,G}(P) = 2^{jn/2} \int_\Omega P(x)\, \Psi^j_{G,m}(x)\, \mathrm{d}x = 0 \quad \text{if } (j,G,m) \in \{JS\}^\Omega. \tag{4.11}$$

Since $|\Omega| < \infty$ it follows that Ω_τ is a bounded domain. Let $\chi_\tau \in D(\Omega_\tau)$ be a cut-off function with $\chi_\tau(x) = 1$ if $x \in \Omega_{2\tau}$. Then (4.6) can be applied to $f = \chi_\tau P \in \overline{F}{}^s_{pq}(\Omega)$ for any $P \in \mathcal{P}^M(\Omega)$. For $\tau = d\,2^{-J}$ with a suitable (small) $d > 0$ one gets by (4.6), (4.5) and (4.11) that

$$P(x) = \sum_{(j,G,m)\in\langle JS\rangle^\Omega} 2^{Jn/2} \left(\int_\Omega P(y)\, \Psi^J_m(y)\, \mathrm{d}y \right) \cdot 2^{-Jn/2}\, \Psi^J_m(x), \tag{4.12}$$

$x \in \Omega_{g2^{-J}}$ for some $g > 0$ which is independent of J. Furthermore, we have by (2.16) that

$$\operatorname{supp} \Psi^J_m \subset Q_{Jr} \subset \Omega_{\varrho 2^{-J}}$$

for some $\varrho > 0$, $r = r(m)$ and $(j,G,m) \in \langle JS\rangle^\Omega$ (then $j = J$). In the next step we prove that there are points $\{x^{k,J,m}\}_{k=1}^K \subset Q^{-1}_{Jr}$ (where the latter is a cube concentric with Q^0_{Jr} with side-length 2^{-J-1}), having pairwise distance of at least $c2^{-J}$ for some $c > 0$, and constants $c_k^{J,m}$ with $|c_k^{J,m}| \leqslant C$ for some $C > 0$ and all admitted J, k, m such that

$$2^{Jn/2} \int_\Omega P(y)\, \Psi^J_m(y)\, \mathrm{d}y = \sum_{k=1}^K c_k^{J,m}\, P(x^{k,J,m}), \quad P \in \mathcal{P}^M(\Omega). \tag{4.13}$$

Taking this for granted we put now for $\tau \sim 2^{-J}$,

$$\begin{aligned} U_\tau f &= \sum_{(j,G,m)\in\langle JS\rangle^\Omega} \sum_{k=1}^K c_k^{J,m}\, f(x^{k,J,m})\, 2^{-Jn/2}\, \Psi^J_m(x) \\ &= \sum_l f(x^l)\, h_l^\tau, \quad f \in \overline{F}{}^s_{pq}(\Omega). \end{aligned} \tag{4.14}$$

Recall that $\overline{F}{}^s_{pq}(\Omega) \hookrightarrow \overline{C}(\Omega)$ since $s > n/p$. Hence (4.14) makes sense. By (4.12) one has (4.10) and also (4.8), where $x^l \Longleftrightarrow x^{k,J,m}$ have the desired properties.

Step 2: It remains to prove (4.13). First, we deal with the one-dimensional case and $\tau \sim 1$. Let

$$P(x) = \sum_{m=0}^{M-1} a_m x^m, \quad 0 < x < 1. \tag{4.15}$$

Let $x_l = l/M$ (or nearby) with $l = 0, \ldots, M-1$. Then the determinant of the M linear algebraic equations for a_m,

$$\sum_{m=0}^{M-1} a_m x_l^m = P(x_l), \quad l = 0, \ldots, M-1, \tag{4.16}$$

is $\prod_{l<r}(x_l - x_r)$ (Vandermonde's determinant). Hence a_m can be expressed by a linear combination of $P(x_l)$ with controllable coefficients,

$$a_m = L_m^M \left(P(x_l)\right), \quad m = 0, \ldots, M-1, \tag{4.17}$$

where M indicates the number of terms. With $n = 1$, $J = 0$, and $\Psi_m^J(y) \sim H(y)$ and, say, $\Omega = (0, 1)$ one gets by

$$\int_0^1 P(x)\, H(x)\, \mathrm{d}x = \sum_{m=0}^{M-1} a_m \int_0^1 x^m\, H(x)\, \mathrm{d}x = L^M \left(P(x^l)\right) \tag{4.18}$$

the desired assertion. If $\tau \sim 2^{-J}$ then one restricts (4.15) to $0 < x < \tau$. One may choose $x_l = \tau l/M$ and with $x_l = \tau y_l$ one gets by (4.16) that

$$\sum_{m=0}^{M-1} a_m \tau^m y_l^m = P(\tau y_l) = P(x_l),$$

which results in

$$\tau^m a_m = L_m^M \left(P(x_l)\right), \quad m = 0, \ldots, M-1, \tag{4.19}$$

where the right-hand side is the same linear combination as in (4.17) which is independent of τ. The counterpart of (4.18) is now given by

$$\begin{aligned} 2^J \int_0^\tau P(x)\, H^J(x)\, \mathrm{d}x &= 2^J \sum_{m=0}^{M-1} a_m \int_0^\tau x^m H^J(x)\, \mathrm{d}x \\ &= \sum_{m=0}^{M-1} \tau^m a_m c_m^J = L^{M,J}\left(P(x_l)\right), \end{aligned}$$

where H^J are uniformly bounded functions (with respect to J) and c_m^J are uniformly bounded coefficients resulting from (4.19). This proves (4.13) for $n = 1$ and $\tau \sim 2^{-J}$.

Step 3: Let $n \geqslant 2$ and $\tau \sim 1$, hence $J = 0$ in (4.13), and, say, $\Omega = \{y : |y| < 1\}$. We ask for the counterpart of (4.17) for

$$P(x) = \sum_{|\gamma| \leqslant M-1} a_\gamma x^\gamma = \sum_{m=0}^{M-1} p_m(x')\, x_n^m, \quad |x| \leqslant 1, \tag{4.20}$$

where $x = (x', x_n)$. By (4.17) we have

$$p_m(x') = L_m^M \left(P(x', x_{n,l})\right), \quad m = 0, \ldots, M-1.$$

For fixed $x_{n,l}$ one has polynomials $P_l(x') = P(x', x_{n,l})$ of degree less than M of $n-1$ variables. By induction we assume that we have for the respective coefficients a counterpart of (4.17) as a linear combination of M^{n-1} terms with controllable coefficients. Inserted in (4.20) one gets the desired n-dimensional version of (4.17) as a linear combination of M^n terms. Afterwards one gets a counterpart of (4.18). The scaling argument is the same as at the end of Step 2. This proves finally (4.13). $\square$

Remark 20. Polynomial reproducing formulas play a role in numerical analysis if one wishes to measure the accuracy of diverse approximations. In connection with sampling numbers (subject of the next section) we relied in [12,20, Section 4.3] on corresponding assertions in [23]. The arguments in [23] are not based on wavelets. On the other hand, it is well known that polynomial reproducing formulas play also a role in wavelet theory, especially in connection with wavelet bases for function spaces on intervals, rectangles, etc. We refer to the literature mentioned in the Introduction, in particular to [1, Section 2.12]. One could introduce $U_\tau f$ in (4.9) for all $f \in \overline{C}(\Omega)$ instead of $f \in \overline{F}{}^s_{pq}(\Omega) \subset \overline{C}(\Omega)$ leaving out $\overline{F}{}^s_{pq}(\Omega)$ in the formulation of the theorem. Then the theorem looks more handsome. But this has no influence on the main assertion (4.10), and one must start the proof explaining the technicalities about $\overline{F}{}^s_{pq}(\Omega)$ (which we incorporated in the formulation of the theorem).

5. Sampling numbers

5.1. *Definitions*

In [12,19] and, based on these papers, in [20, Section 4.3] we dealt with sampling numbers of compact embeddings between function spaces $B^s_{pq}(\Omega)$, $F^s_{pq}(\Omega)$, and $L_t(\Omega)$ on bounded Lipschitz domains in $\mathbb{R}^n$. We continue these studies now for the spaces and domains considered above. Otherwise we use the same techniques as in [12] prepared by the considerations above in particular by the polynomial reproducing formulas. The relevant literature may be found in [12,19,20] which will not be repeated here. But we recall some basic definitions.

Let Ω be an arbitrary domain in $\mathbb{R}^n$ with $|\Omega| < \infty$. Let $G_1(\Omega)$ be either $\overline{F}{}^s_{pq}(\Omega)$ according to Definition 2 with

$$0 < p \leqslant \infty, \quad 0 < q \leqslant \infty, \quad s > \max\left(\frac{n}{p}, \sigma_{pq}\right)$$

($q = \infty$ if $p = \infty$) or $\widetilde{B}{}^s_{pq}(\Omega)$, $\widetilde{F}{}^s_{pq}(\Omega)$ according to Definition 15(iii) with

$$0 < p \leqslant \infty, \quad 0 < q \leqslant \infty, \quad s > n/p$$

($p < \infty$ for the F-spaces). Recall that all these spaces are continuously embedded in $\overline{C}(\Omega)$, where $\overline{C}(\Omega)$ has the same meaning as at the beginning of Section 4.2. Since $|\Omega| < \infty$ one has also a continuous embedding in $L_t(\Omega)$, $0 < t \leqslant \infty$. Let

$$G_2(\Omega) = \overline{C}(\Omega) \quad \text{or} \quad G_2(\Omega) = L_t(\Omega), \quad 0 < t \leqslant \infty. \tag{5.1}$$

Then one gets as a by-product of the considerations below that

$$\mathrm{id}_\Omega : \quad G_1(\Omega) \hookrightarrow G_2(\Omega) \tag{5.2}$$

is not only continuous but also compact. As for technicalities connected with these embeddings one may consult [12,20, Section 4.3.1] including the explanations and references given there. This will not be repeated here. In any case by (5.1), (5.2), pointwise evaluation of $f \in G_1(\Omega)$ makes sense. Let $\{x^k\}_{k=1}^K \subset \Omega$. Then the *information map*

$$N_K : \quad G_1(\Omega) \mapsto \mathbb{C}^K, \quad K \in \mathbb{N}, \tag{5.3}$$

given by

$$N_K f = \left(f(x^1), \ldots, f(x^K) \right), \quad f \in G_1(\Omega), \tag{5.4}$$

is reasonable. Let U_K,

$$U_K = \Phi_K \circ N_K \quad \text{where } \Phi_K : \quad \mathbb{C}^K \mapsto G_2(\Omega) \tag{5.5}$$

is an arbitrary map (also called *method* or *algorithm*). Hence

$$U_K f = \Phi_K \left(f(x^1), \ldots, f(x^K) \right) \in G_2(\Omega), \quad f \in G_1(\Omega).$$

Definition 21. Let Ω be an arbitrary domain in $\mathbb{R}^n$ with $|\Omega| < \infty$. Let $G_1(\Omega)$ and $G_2(\Omega)$ be the above spaces and let id_Ω be the embedding (5.2).

(i) Then

$$g_K(\mathrm{id}_\Omega) = \inf \left[\sup_{\|f \,|G_1(\Omega)\| \leqslant 1} \|f - U_K f \,|G_2(\Omega)\| \right] \tag{5.6}$$

is the Kth sampling number where the infimum is taken over all K-tuples $\{x^k\}_{k=1}^K \subset \Omega$ and all maps U_K according to (5.3)–(5.5).

(ii) The linear sampling numbers $g_K^{\mathrm{lin}}(\mathrm{id}_\Omega)$ are given by (5.6) where the infimum is taken over all K-tuples $\{x^k\}_{k=1}^K$ and all linear maps U_K with

$$U_K f = \sum_{k=1}^K f(x^k)\, h_k, \quad h_k \in G_2(\Omega), \quad f \in G_1(\Omega). \tag{5.7}$$

Remark 22. This is an adapted version of [12, Definition 17, 20, Definition 4.32]. There we dealt with bounded Lipschitz domains. If one admits in (5.6) not only the specific linear maps in (5.7) but all linear maps from $G_2(\Omega)$ into $G_1(\Omega)$ with rank less than $K+1$ then one gets the well-known approximation numbers $a_{K+1}(\mathrm{id}_\Omega)$, hence

$$a_{K+1}(\mathrm{id}_\Omega) \leqslant g_K^{\mathrm{lin}}(\mathrm{id}_\Omega), \quad K \in \mathbb{N}.$$

According to Theorem 23 below in all cases considered $g_K^{\mathrm{lin}}(\mathrm{id}_\Omega)$ tends to zero if $K \to \infty$. Then one has the same assertion for the approximation numbers $a_K(\mathrm{id}_\Omega)$ with the well-known consequence that id_Ω is compact.

5.2. *Main assertions*

After all these preparations we are now in the position to apply the techniques developed in [12] to determine the behaviour of the sampling numbers of the compact embeddings id_Ω in (5.2) with the specifications indicated. Recall that $a_+ = \max(a, 0)$ if $a \in \mathbb{R}$. As usual, $\sim$ means that there are positive equivalence constants which are independent of $k \in \mathbb{N}$.

Theorem 23. (i) *Let Ω be an arbitrary domain in $\mathbb{R}^n$ with $|\Omega| < \infty$. Let $\overline{F}{}^s_{pq}(\Omega)$ with*

$$0 < p \leqslant \infty, \quad 0 < q \leqslant \infty, \quad s > \max\left(\frac{n}{p},\, \sigma_{pq}\right) \tag{5.8}$$

($q = \infty$ if $p = \infty$) *be the refined localisation spaces according to Definition* 2. *Then*

$$\overline{\mathrm{id}_\Omega} : \quad \overline{F}{}^s_{pq}(\Omega) \hookrightarrow L_t(\Omega), \quad 0 < t \leqslant \infty$$

(*where $L_\infty(\Omega)$ can be replaced by $\overline{C}(\Omega)$*) *is compact. Furthermore*,

$$g_k(\overline{\mathrm{id}_\Omega}) \sim g_k^{\mathrm{lin}}(\overline{\mathrm{id}_\Omega}) \sim k^{-s/n+(1/p-1/t)_+}, \quad k \in \mathbb{N}.$$

(ii) *Let Ω be an E-thick domain in $\mathbb{R}^n$ according to Definition* 12 *with $|\Omega| < \infty$ and let $\widetilde{A}{}^s_{pq}(\Omega)$ with $A = B$ or $A = F$ be the spaces as introduced in Definition* 15(iii) *with*

$$0 < p \leqslant \infty, \quad 0 < q \leqslant \infty, \quad s > \frac{n}{p}$$

($p < \infty$ *for the F-spaces*). *Then*

$$\widetilde{\mathrm{id}_\Omega} : \quad \widetilde{A}{}^s_{pq}(\Omega) \hookrightarrow L_t(\Omega), \quad 0 < t \leqslant \infty$$

(*where $L_\infty(\Omega)$ can be replaced by $\overline{C}(\Omega)$*) *is compact. Furthermore*,

$$g_k(\widetilde{\mathrm{id}_\Omega}) \sim g_k^{\mathrm{lin}}(\widetilde{\mathrm{id}_\Omega}) \sim k^{-s/n+(1/p-1/t)_+}, \quad k \in \mathbb{N}.$$

Proof. *Step* 1: As for the compactness we refer to the comments in Remark 22.

Step 2: We prove (i). Let $p < t \leqslant \infty$. Then

$$\overline{F}{}^s_{pq}(\Omega) \hookrightarrow \overline{F}{}^\sigma_{t\infty}(\Omega), \quad \sigma - \frac{n}{t} = s - \frac{n}{p} > 0. \tag{5.9}$$

This follows from the well-known embedding

$$F^s_{pq}(\mathbb{R}^n) \hookrightarrow F^\sigma_{t\infty}(\mathbb{R}^n),$$

(2.10) and the monotonicity of the ℓ_r-spaces. We wish to prove that

$$g_k^{\mathrm{lin}}(\overline{\mathrm{id}_\Omega}) \leqslant c\, k^{-s/n+(1/p-1/t)_+}, \quad k \in \mathbb{N}, \tag{5.10}$$

in all cases. Using (5.9) if $p < t$, and Hölder's inequality for $L_t(\Omega)$ if $t < p$ based on $|\Omega| < \infty$, it follows that we may assume $p = t$, which means

$$g_k^{\mathrm{lin}}(\overline{\mathrm{id}_\Omega}) \leqslant c\, k^{-s/n}, \quad k \in \mathbb{N}, \tag{5.11}$$

where

$$\overline{\mathrm{id}_\Omega} : \quad \overline{F}{}^s_{pq}(\Omega) \hookrightarrow L_p(\Omega), \quad 0 < p \leqslant \infty$$

($q=\infty$ if $p=\infty$). Let Ω_τ with $0<\tau<1$ as in (4.7) and let χ_τ be the same cut-off function as after (4.11) with the usual conditions for $D^\gamma \chi_\tau$ such that the pointwise multiplier assertion in Proposition 42(ii) can be applied uniformly with respect to τ. In particular, one gets for $f^\tau = (1-\chi_\tau) f$ with $f \in \overline{F}{}^s_{pq}(\Omega)$ that

$$\| f^\tau | L_p(\Omega) \| \leqslant c\, \tau^s \, \| \delta^{-s}(\cdot) f \, | L_p(\Omega) \| \leqslant c\, \tau^s \, \| f \, | \overline{F}{}^s_{pq}(\Omega) \|, \tag{5.12}$$

where we used (2.14). Next we apply the polynomial reproducing formula (4.9), (4.10) to $f_\tau = \chi_\tau f$. But then one is precisely in the same situation as in [12, Proposition 21 and its proof, 20, Proposition 4.36] where one has to use now Theorem 4. This will not be repeated here. With $\tau \sim 2^{-J}$ and $k \sim 2^{Jn}$ one gets (5.11) from [12,20] applied (uniformly) to f_τ and from (5.12). This proves (5.10). The rest is now the same as in [12,20]. In particular the estimate from below

$$c\, k^{-s/n+(1/p-1/t)_+} \leqslant g_k(\overline{\mathrm{id}_\Omega}) \leqslant g_k^{\mathrm{lin}}(\overline{\mathrm{id}_\Omega}), \quad k \in \mathbb{N},$$

for some $c>0$ is local and can be taken over verbally.

Step 3: We prove part (ii). By Corollary 18 and part (i) we have the desired assertion for the spaces $\widetilde{F}{}^s_{pq}(\Omega)$ with (5.8). Let $0<q\leqslant\infty$ and

$$0 < p \leqslant \infty, \quad s_0 > s > s_1 > n/p, \quad s = (1-\theta)s_0 + \theta s_1.$$

The well-known real interpolation formula in $\mathbb{R}^n$,

$$B^s_{pq}(\mathbb{R}^n) = \left(F^{s_0}_{pp}(\mathbb{R}^n), F^{s_1}_{pp}(\mathbb{R}^n) \right)_{\theta,q}$$

has the Ω-counterpart

$$\widetilde{B}{}^s_{pq}(\Omega) = \left(\widetilde{F}{}^{s_0}_{pp}(\Omega), \widetilde{F}{}^{s_1}_{pp}(\Omega) \right)_{\theta,q}. \tag{5.13}$$

This is not obvious and requires some efforts. But we omit the details and take it for granted. For the same linear operator U_τ according to (4.9) we get by the above considerations

$$\| f - U_\tau f \, | L_t(\Omega) \| \leqslant c \tau^{s_0 - n(1/p-1/t)_+} \| f \, | \widetilde{F}{}^{s_0}_{pp}(\Omega) \|, \quad f \in \widetilde{F}{}^{s_0}_{pp}(\Omega), \tag{5.14}$$

and

$$\| f - U_\tau f \, | L_t(\Omega) \| \leqslant c \tau^{s_1 - n(1/p-1/t)_+} \| f \, | \widetilde{F}{}^{s_1}_{pp}(\Omega) \|, \quad f \in \widetilde{F}{}^{s_1}_{pp}(\Omega). \tag{5.15}$$

One may also consult [20, (4.188)]. Then one gets by (5.13)–(5.15) and the interpolation property that

$$\| f - U_\tau f \, | L_t(\Omega) \| \leqslant c\, \tau^{s - n(1/p-1/t)_+} \| f \, | \widetilde{B}{}^s_{pq}(\Omega) \|$$

and, using elementary embedding,

$$\| f - U_\tau f \, | L_t(\Omega) \| \leqslant c \tau^{s - n(1/p-1/t)_+} \| f \, | \widetilde{F}{}^s_{pq}(\Omega) \|.$$

This proves the counterpart of (5.10),

$$g_k^{\mathrm{lin}}(\widetilde{\mathrm{id}_\Omega}) \leqslant c\, k^{-s/n+(1/p-1/t)_+}, \qquad k \in \mathbb{N}.$$

The rest is now the same as in Step 2 and as in [12,20]. $\square$

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Appendix A. Function spaces on Euclidean *n*-space

A.1. Definitions

We use standard notation. Let $\mathbb{N}$ be the collection of all natural numbers and $\mathbb{N}_0 = \mathbb{N} \cup \{0\}$. Let $\mathbb{R}^n$ be Euclidean n-space, where $n \in \mathbb{N}$. Put $\mathbb{R} = \mathbb{R}^1$, whereas $\mathbb{C}$ is the complex plane. Let $S(\mathbb{R}^n)$ be the usual Schwartz space and $S'(\mathbb{R}^n)$ the space of all tempered distributions on $\mathbb{R}^n$. Furthermore, $L_p(\mathbb{R}^n)$ with $0 < p \leqslant \infty$, is the standard quasi-Banach space with respect to the Lebesgue measure, quasi-normed by

$$\|f\,|L_p(\mathbb{R}^n)\| = \left(\int_{\mathbb{R}^n} |f(x)|^p \,\mathrm{d}x\right)^{1/p}$$

with the obvious modification if $p = \infty$.

As usual, $\mathbb{Z}$ is the collection of all integers; and $\mathbb{Z}^n$ where $n \in \mathbb{N}$, denotes the lattice of all points $m = (m_1, \ldots, m_n) \in \mathbb{R}^n$ with $m_j \in \mathbb{Z}$. Let $\mathbb{N}_0^n$, where $n \in \mathbb{N}$, be the set of all multi-indices,

$$\alpha = (\alpha_1, \ldots, \alpha_n) \quad \text{with } \alpha_j \in \mathbb{N}_0 \text{ and } |\alpha| = \sum_{j=1}^{n} \alpha_j.$$

If $x = (x_1, \ldots, x_n) \in \mathbb{R}^n$ and $\beta = (\beta_1, \ldots, \beta_n) \in \mathbb{N}_0^n$ then we put

$$x^\beta = x^{\beta_1} \cdots x_n^{\beta_n} \quad \text{(monomials)}.$$

If $\varphi \in S(\mathbb{R}^n)$ then

$$\widehat{\varphi}(\xi) = (F\varphi)(\xi) = (2\pi)^{-n/2} \int_{\mathbb{R}^n} e^{-ix\xi} \varphi(x)\,\mathrm{d}x, \quad \xi \in \mathbb{R}^n, \tag{A.1}$$

denotes the Fourier transform of φ. As usual, $F^{-1}\varphi$ or $\varphi^\vee$, stands for the inverse Fourier transform, given by the right-hand side of (A.1) with i in place of $-i$. Here, $x\xi$ denotes the scalar product in $\mathbb{R}^n$. Both F and F^{-1} are extended to $S'(\mathbb{R}^n)$ in the standard way. Let $\varphi_0 \in S(\mathbb{R}^n)$ with

$$\varphi_0(x) = 1 \quad \text{if } |x| \leqslant 1 \text{ and } \varphi_0(y) = 0 \quad \text{if } |y| \geqslant 3/2,$$

and let

$$\varphi_k(x) = \varphi_0(2^{-k}x) - \varphi_0(2^{-k+1}x), \quad x \in \mathbb{R}^n, \; k \in \mathbb{N}.$$

Since $\sum_{j=0}^{\infty} \varphi_j(x) = 1$ for $x \in \mathbb{R}^n$, the φ_j form a dyadic resolution of unity. The entire analytic functions $\left(\varphi_j \widehat{f}\right)^\vee(x)$ make sense pointwise for any $f \in S'(\mathbb{R}^n)$.

Definition 24. Let $\varphi = \{\varphi_j\}_{j=0}^{\infty}$ be the above dyadic resolution of unity.

(i) Let

$$0 < p \leqslant \infty, \quad 0 < q \leqslant \infty, \quad s \in \mathbb{R}.$$

Then $B^s_{pq}(\mathbb{R}^n)$ is the collection of all $f \in S'(\mathbb{R}^n)$ such that

$$\|f\,|B^s_{pq}(\mathbb{R}^n)\|_\varphi = \left(\sum_{j=0}^{\infty} 2^{jsq}\,\|\left(\varphi_j \widehat{f}\right)^\vee |L_p(\mathbb{R}^n)\|^q\right)^{1/q} < \infty \tag{A.2}$$

(with the usual modification if $q = \infty$).

(ii) Let

$$0 < p < \infty, \quad 0 < q \leqslant \infty, \quad s \in \mathbb{R}.$$

Then $F^s_{pq}(\mathbb{R}^n)$ is the collection of all $f \in S'(\mathbb{R}^n)$ such that

$$\|f\,|F^s_{pq}(\mathbb{R}^n)\|_\varphi = \left\| \left(\sum_{j=0}^{\infty} 2^{jsq}\,|\left(\varphi_j \widehat{f}\right)^\vee(\cdot)|^q\right)^{1/q} \Bigg| L_p(\mathbb{R}^n)\right\| < \infty \tag{A.3}$$

(with the usual modification if $q = \infty$).

Remark 25. The theory of these spaces may be found in [15,16,20], including many historical references. We only mention that these spaces are independent of φ (equivalent quasi-norms for admitted φ's). This justifies our omission of the subscript φ in (A.2), (A.3) in the sequel. In Section 2.1 we listed some (more or less classical) special cases.

A.2. Properties

We collect those (and only those) properties needed in the main body of this paper and from which we believe that they complement in a decisive way classical instruments such as derivatives and differences in connection with problems as treated here.

A.2.1. Atoms

Let Q_{jm} be cubes in $\mathbb{R}^n$ with sides parallel to the axes of coordinates, centred at $2^{-j}m$ with side length 2^{-j+1} where $m \in \mathbb{Z}^n$ and $j \in \mathbb{N}_0$. If Q is a cube in $\mathbb{R}^n$ and $r > 0$ then rQ is the cube in $\mathbb{R}^n$ concentric with Q and with side-length r times of the side length of Q. Let χ_{jm} be the characteristic function of Q_{jm}.

Definition 26. Let $0 < p \leqslant \infty$, $0 < q \leqslant \infty$. Then b_{pq} is the collection of all sequences

$$\lambda = \left\{\lambda_{jm} \in \mathbb{C} : \ j \in \mathbb{N}_0,\ m \in \mathbb{Z}^n\right\} \tag{A.4}$$

such that

$$\|\lambda\,|b_{pq}\| = \left(\sum_{j=0}^{\infty}\left(\sum_{m\in\mathbb{Z}^n} |\lambda_{jm}|^p\right)^{q/p}\right)^{1/q} < \infty,$$

and f_{pq} is the collection of all sequences λ according to (A.4) such that

$$\|\lambda|f_{pq}\| = \left\| \left(\sum_{j,m} 2^{jnq/p}\,|\lambda_{jm}\,\chi_{jm}(\cdot)|^q\right)^{1/q} \Bigg| L_p(\mathbb{R}^n)\right\| < \infty. \tag{A.5}$$

Remark 27. If $p = \infty$ and/or $q = \infty$ then one has to modify in the usual way. Note that the factor $2^{jnq/p}$ in (A.5) disappears if one relies on the p-normalised characteristic function $\chi_{jm}^{(p)}(x) = 2^{jn/p}\chi_{jm}(x)$. Next we introduce atoms, which may be discontinuous.

Definition 28. Let $s \in \mathbb{R}$, $0 < p \leqslant \infty$, $K \in \mathbb{N}_0$, $L \in \mathbb{N}_0$, and $c \geqslant 1$. Then L_∞-functions $a_{jm} : \mathbb{R}^n \to \mathbb{C}$ with $j \in \mathbb{N}_0$, $m \in \mathbb{Z}^n$, are called (s, p)-atoms if

$$\operatorname{supp} a_{jm} \subset c\, Q_{jm}, \quad j \in \mathbb{N}_0, \quad m \in \mathbb{Z}^n,$$

there exists all (classical) derivatives $D^\alpha a_{jm}$ with $|\alpha| \leqslant K$ such that

$$|D^\alpha a_{jm}(x)| \leqslant 2^{-j(s-n/p)+j|\alpha|}, \quad |\alpha| \leqslant K, \quad j \in \mathbb{N}_0, \quad m \in \mathbb{Z}^n, \tag{A.6}$$

and

$$\int_{\mathbb{R}^n} x^\beta a_{jm}(x)\, \mathrm{d}x = 0 \quad \text{for } |\beta| < L, \quad j \in \mathbb{N}, \quad m \in \mathbb{Z}^n. \tag{A.7}$$

Remark 29. There are no moment conditions (A.7) for $a_{0,m}$. Furthermore, if $L = 0$ then (A.7) is empty (no conditions). Of course, the above atoms depend on K, L, and c. But this will not be indicated. We put as usual

$$\sigma_p = n\left(\frac{1}{p} - 1\right)_+ \quad \text{and} \quad \sigma_{pq} = n\left(\frac{1}{\min(p, q)} - 1\right)_+ \tag{A.8}$$

where $b_+ = \max(b, 0)$ if $b \in \mathbb{R}$.

Theorem 30. (i) *Let* $0 < p \leqslant \infty$, $0 < q \leqslant \infty$, $s \in \mathbb{R}$. *Let* $K \in \mathbb{N}_0$, $L \in \mathbb{N}_0$ *with*

$$K > s \quad \text{and} \quad L > \sigma_p - s \tag{A.9}$$

be fixed. Then $f \in S'(\mathbb{R}^n)$ *belongs to* $B^s_{pq}(\mathbb{R}^n)$ *if, and only if, it can be represented as*

$$f = \sum_{j=0}^{\infty} \sum_{m \in \mathbb{Z}^n} \lambda_{jm}\, a_{jm}, \tag{A.10}$$

where for fixed $c \geqslant 1$, a_{jm} *are* (s, p)*-atoms according to Definition* 28 *with* (A.9) *and* $\lambda \in b_{pq}$. *Furthermore*,

$$\| f \,|B^s_{pq}(\mathbb{R}^n)\| \sim \inf \|\lambda \,|b_{pq}\|$$

are equivalent quasi-norms where the infimum is taken over all admissible representations (A.10).

(ii) *Let* $0 < p < \infty$, $0 < q \leqslant \infty$, $s \in \mathbb{R}$. *Let* $K \in \mathbb{N}_0$, $L \in \mathbb{N}_0$ *with*

$$K > s \quad \text{and} \quad L > \sigma_{pq} - s \tag{A.11}$$

be fixed. Then $f \in S'(\mathbb{R}^n)$ *belongs to* $F^s_{pq}(\mathbb{R}^n)$ *if, and only if, it can be represented by* (A.10) *where now for fixed* $c \geqslant 1$, a_{jm} *are* (s, p)*-atoms according to Definition* 28 *with* (A.11) *and* $\lambda \in f_{pq}$. *Furthermore*,

$$\| f \,|F^s_{pq}(\mathbb{R}^n)\| \sim \inf \|\lambda \,|f_{pq}\|$$

are equivalent quasi-norms where the infimum is taken over all admissible representations (A.10).

Remark 31. These formulations coincide essentially with [20, Section 1.5.1]. There one finds technical comments how the convergence in (A.10) must be understood. Atoms of the above type go back essentially to [6,7]. But more details about the history of atoms may be found in [16, Section 1.9].

A.2.2. Local means

Compactly supported kernels of local means are dual to atoms according to Definition 28. The cubes Q_{jm} have the same meaning as above.

Definition 32. Let $A \in \mathbb{N}_0$, $B \in \mathbb{N}_0$ and $C > 0$. Then L_∞-functions

$$k_{jm} : \mathbb{R}^n \to \mathbb{C} \quad \text{with } j \in \mathbb{N}_0, \ \ m \in \mathbb{Z}^n,$$

are called kernels if

$$\operatorname{supp} k_{jm} \subset C\, Q_{jm}, \quad j \in \mathbb{N}_0, \ \ m \in \mathbb{Z}^n,$$

there exist all (classical) derivatives $D^\alpha k_{jm}$ with $|\alpha| \leqslant A$ such that

$$|D^\alpha k_{jm}(x)| \leqslant C\, 2^{jn+j|\alpha|}, \quad |\alpha| \leqslant A, \ j \in \mathbb{N}_0, \ m \in \mathbb{Z}^n, \tag{A.12}$$

and

$$\int_{\mathbb{R}^n} x^\beta\, k_{jm}(x)\,\mathrm{d}x = 0 \quad \text{for } |\beta| < B, \ j \in \mathbb{N}, \ m \in \mathbb{Z}^n. \tag{A.13}$$

Remark 33. There are no moment conditions (A.13) for $k_{0,m}$. If $B = 0$ then (A.13) is empty. Compared with Definition 28 for atoms we have different normalisations in (A.6) and (A.12) (also due to the history of atoms). Roughly speaking atoms are normalised building blocks in $B^s_{pq}(\mathbb{R}^n)$ and $F^s_{pq}(\mathbb{R}^n)$, reflected by Theorem 30 based on the sequence spaces b_{pq} and f_{pq} in Definition 26. Now we adapt these sequence spaces to the above kernels. Again χ_{jm} are the characteristic functions of Q_{jm}.

Definition 34. Let $s \in \mathbb{R}$, $0 < p \leqslant \infty$, $0 < q \leqslant \infty$. Then $\overline{b}^s_{pq}$ is the collection of all sequences λ according to (A.4) such that

$$\|\lambda\,|\overline{b}^s_{pq}\| = \left(\sum_{j=0}^{\infty} 2^{j(s-n/p)q} \left(\sum_{m \in \mathbb{Z}^n} |\lambda_{jm}|^p \right)^{q/p} \right)^{1/q} < \infty,$$

and $\overline{f}^s_{pq}$ is the collection of all sequences λ according to (A.4) such that

$$\|\lambda\,|\overline{f}^s_{pq}\| = \left\| \left(\sum_{j,m} 2^{jsq}\,|\lambda_{jm}\,\chi_{jm}(\cdot)|^q \right)^{1/q} \middle| L_p(\mathbb{R}^n) \right\| < \infty.$$

Remark 35. In connection with wavelets we introduce the slightly modified sequence spaces b^s_{pq} and f^s_{pq} without the above bar. Otherwise we wish to specify the sequence λ in (A.4) by the sequence of local means

$$k(f) = \left\{ k_{jm}(f) : \ j \in \mathbb{N}_0, \ m \in \mathbb{Z}^n \right\}, \tag{A.14}$$

where

$$k_{jm}(f) = \int_{\mathbb{R}^n} k_{jm}(y)\, f(y)\, \mathrm{d}y = \big(f, k_{jm}\big) \tag{A.15}$$

considered as a dual pairing where $f \in B^s_{pq}(\mathbb{R}^n)$ or $f \in F^s_{pq}(\mathbb{R}^n)$. This requires that k_{jm} belongs to the dual spaces of $B^s_{pq}(\mathbb{R}^n)$ or $F^s_{pq}(\mathbb{R}^n)$. This will always be the case in what follows. But we do not discuss this point here.

Theorem 36. (i) *Let* $0 < p \leqslant \infty$, $0 < q \leqslant \infty$, $s \in \mathbb{R}$. *Let* k_{jm} *be kernels according to Definition* 32 *where* $A \in \mathbb{N}_0$, $B \in \mathbb{N}_0$ *with*

$$A > \sigma_p - s, \quad B > s, \tag{A.16}$$

and $C > 0$ *be fixed. Let* $k(f)$ *be as in* (A.14), (A.15). *Then for some* $c > 0$ *and all* $f \in B^s_{pq}(\mathbb{R}^n)$,

$$\|k(f)\, |\bar{b}^s_{pq}\| \leqslant c\, \|f\, |B^s_{pq}(\mathbb{R}^n)\|.$$

(ii) *Let* $0 < p < \infty$, $0 < q \leqslant \infty$, $s \in \mathbb{R}$. *Let* k_{jm} *and* $k(f)$ *be again the above kernels where* $A \in \mathbb{N}_0$, $B \in \mathbb{N}_0$, *with*

$$A > \sigma_{pq} - s, \quad B > s, \tag{A.17}$$

and $C > 0$ *are fixed. Then for some* $c > 0$ *and all* $f \in F^s_{pq}(\mathbb{R}^n)$,

$$\|k(f)\, |\bar{f}^s_{pq}\| \leqslant c\, \|f\, |F^s_{pq}(\mathbb{R}^n)\|.$$

Remark 37. The duality between atoms and kernels is well reflected by (A.9), (A.11) compared with (A.16), (A.17). Later on we will even choose

$$K = B \quad \text{and} \quad L = A,$$

changing the roles of the needed smoothness and cancellations. The proof of the above theorem is somewhat complicated and will be shifted to a later occasion. But local means (of continuous, or, as above, of discrete type) are well known and have their own history. This theory started (at least as far as presentations in books are concerned) in [16, Sections 1.8.4, 2.4.6, 2.5.3]. A recent account may be found in [20, Section 1.4]. On the other hand dual pairings of type (A.15) have been considered constantly in the theory of the spaces $B^s_{pq}(\mathbb{R}^n)$ and $F^s_{pq}(\mathbb{R}^n)$ at many occasions. Nearest to us and to the above theorem might be [9].

A.2.3. Wavelets

We suppose that the reader is familiar with wavelets in $\mathbb{R}^n$ of Daubechies type and the related multi-resolution analysis. The standard references are [5,10,11,24]. A short summary of what is needed in our context may also be found in [20, Section 1.7.3]. In [20, Section 3.1] we dealt with wavelet bases and wavelet isomorphisms for all spaces $B^s_{pq}(\mathbb{R}^n)$ and $F^s_{pq}(\mathbb{R}^n)$. We describe now an improved version based on the new Theorem 36 which was not known to us when [20, Section 3.1] was written. This improvement is helpful even in $\mathbb{R}^n$, but indispensable when it comes to domains.

As usual $C^u(\mathbb{R})$ collects all (complex-valued) continuous functions on $\mathbb{R}$ having continuous bounded derivatives up to order $u \in \mathbb{N}$. Let

$$\psi_F \in C^u(\mathbb{R}), \quad \psi_M \in C^u(\mathbb{R}), \quad u \in \mathbb{N}, \tag{A.18}$$

be real compactly supported Daubechies wavelets with

$$\int_{\mathbb{R}} \psi_M(x)\, x^v \,\mathrm{d}x = 0 \quad \text{for } v \in \mathbb{N}_0 \quad \text{with } v < u. \tag{A.19}$$

Recall that ψ_F is called the scaling function (father wavelet) and ψ_M is the associated (mother) wavelet. We extend these wavelets from $\mathbb{R}$ to $\mathbb{R}^n$ by the usual tensor procedure. Let

$$G = (G_1, \ldots, G_n) \in G^0 = \{F, M\}^n \tag{A.20}$$

if G_r is either F or M. Let

$$G = (G_1, \ldots, G_n) \in G^j \in \{F, M\}^{n*}, \quad j \in \mathbb{N}, \tag{A.21}$$

if G_r is either F or M and where $*$ indicates that at least one of the components of G must be an M. Hence G^0 has 2^n elements, whereas G^j with $j \in \mathbb{N}$ has $2^n - 1$ elements. Let

$$\Psi^j_{G,m}(x) = 2^{jn/2} \prod_{r=1}^{n} \psi_{G_r}(2^j x_r - m_r), \quad G \in G^j, \;\; m \in \mathbb{Z}^n, \tag{A.22}$$

where $j \in \mathbb{N}_0$. We always assume that ψ_F and ψ_M in (A.18) have L_2-norm 1. Then

$$\left\{ \Psi^j_{G,m} : \; j \in \mathbb{N}_0, \; G \in G^j, \; m \in \mathbb{Z}^n \right\} \tag{A.23}$$

is an orthonormal basis in $L_2(\mathbb{R}^n)$ (for any $u \in \mathbb{N}$) and

$$f = \sum_{j=0}^{\infty} \sum_{G \in G^j} \sum_{m \in \mathbb{Z}^n} \lambda^{j,G}_m \, 2^{-jn/2} \, \Psi^j_{G,m} \tag{A.24}$$

with

$$\lambda^{j,G}_m = \lambda^{j,G}_m(f) = 2^{jn/2} \int_{\mathbb{R}^n} f(x)\, \Psi^j_{G,m}(x) \,\mathrm{d}x = 2^{jn/2} \left(f, \Psi^j_{G,m} \right) \tag{A.25}$$

is the corresponding expansion, adapted to our later needs, where $2^{-jn/2}\Psi^j_{G,m}$ are uniformly bounded functions. One may ask whether (A.23) remains to be an (unconditional) basis in other spaces. First candidates are $L_p(\mathbb{R}^n)$ with $1 < p < \infty$ but also related (fractional) Sobolev spaces and classical Besov spaces. We refer to the books mentioned at the beginning of this Section A.2.3 and to [20, Remarks 1.63, 1.66] for more details and further references. An extension of this theory to all spaces $B^s_{pq}(\mathbb{R}^n)$ and $F^s_{pq}(\mathbb{R}^n)$ has been given in [9,18,20, Section 3.1.3, Theorem 3.5]. We describe an improved version of [18,20, Theorem 3.5]. For this purpose we adapt the sequence spaces introduced in Definition 34 (now without the bar).

Definition 38. Let $s \in \mathbb{R}$, $0 < p \leqslant \infty$, $0 < q \leqslant \infty$. Then b^s_{pq} is the collection of all sequences

$$\lambda = \left\{ \lambda^{j,G}_m \in \mathbb{C} : \; j \in \mathbb{N}_0, \; G \in G^j, \; m \in \mathbb{Z}^n \right\} \tag{A.26}$$

such that

$$\| \lambda \,|b^s_{pq} \| = \left(\sum_{j=0}^{\infty} 2^{j(s-n/p)q} \sum_{G \in G^j} \left(\sum_{m \in \mathbb{Z}^n} |\lambda^{j,G}_m|^p \right)^{q/p} \right)^{1/q} < \infty \tag{A.27}$$

and f^s_{pq} is the collection of all sequences (A.26) such that

$$\|\lambda \,|\, f^s_{pq}\| = \left\| \left(\sum_{j,G,m} 2^{jsq} \, |\lambda^{j,G}_m \, \chi_{jm}(\cdot)|^q \right)^{1/q} \right| L_p(\mathbb{R}^n) \Big\| < \infty \tag{A.28}$$

(obviously modified if $p = \infty$ and/or $q = \infty$).

Remark 39. Compared with Definition 34 we have now the additional summation over G in agreement with (A.24). Of course the summation over j, G, m in (A.28) is the same as in (A.27).

Theorem 40. (i) *Let* $0 < p \leqslant \infty$, $0 < q \leqslant \infty$, $s \in \mathbb{R}$, *and*

$$u > \max(s, \sigma_p - s) \tag{A.29}$$

in (A.18), (A.19). *Then* $f \in S'(\mathbb{R}^n)$ *is an element of* $B^s_{pq}(\mathbb{R}^n)$ *if, and only if, it can be represented by* (A.24) *with* $\lambda \in b^s_{pq}$. *Furthermore, if* $f \in B^s_{pq}(\mathbb{R}^n)$ *then representation* (A.24) *is unique with* $\lambda = \lambda(f)$ *according to* (A.25) *and* $f \mapsto \lambda(f)$ *is an isomorphic map of* $B^s_{pq}(\mathbb{R}^n)$ *onto* b^s_{pq}.

(ii) *Let* $0 < p < \infty$, $0 < q \leqslant \infty$, $s \in \mathbb{R}$, *and*

$$u > \max(s, \sigma_{pq} - s) \tag{A.30}$$

in (A.18), (A.19). *Then* $f \in S'(\mathbb{R}^n)$ is *an element of* $F^s_{pq}(\mathbb{R}^n)$ *if, and only if, it can be represented by* (A.24) *with* $\lambda \in f^s_{pq}$. *Furthermore, if* $f \in F^s_{pq}(\mathbb{R}^n)$ *then the representation* (A.24) *is unique with* $\lambda = \lambda(f)$ *according to* (A.25) *and* $f \mapsto \lambda(f)$ *is an isomorphic map of* $F^s_{pq}(\mathbb{R}^n)$ *onto* f^s_{pq}.

Remark 41. Compared with [18,20, Theorem 3.5] we have now better and more natural conditions (A.29), (A.30) for u in (A.18), (A.19). In [20] we relied on the characterisation of $B^s_{pq}(\mathbb{R}^n)$ and $F^s_{pq}(\mathbb{R}^n)$ according to [20, Propositon 3.3] in terms of maximal functions. This spoils the estimate for u. Replacing this proposition by the above Theorem 36 then one gets without any additional efforts the above theorem. If $p < \infty, q < \infty$ then (A.23) is an unconditional Schauder basis in $B^s_{pq}(\mathbb{R}^n)$ and $F^s_{pq}(\mathbb{R}^n)$. Otherwise we refer to [20] for a more careful discussion of convergence.

A.2.4. Homogeneity and pointwise multipliers

We need a few further preparations for the spaces $F^s_{pq}(\mathbb{R}^n)$ with $p < \infty$ complemented now by $F^s_{\infty\infty}(\mathbb{R}^n) = B^s_{\infty\infty}(\mathbb{R}^n)$.

Proposition 42. *Let*

$$0 < p \leqslant \infty, \quad 0 < q \leqslant \infty, \quad s > \sigma_{pq}$$

(*with* $q = \infty$ if $p = \infty$) *and* $0 < \varepsilon \leqslant 1$.

(i) *Then*

$$\|f(\varepsilon \cdot) \,|\, F^s_{pq}(\mathbb{R}^n)\| \sim \varepsilon^{s-n/p} \|f \,|\, F^s_{pq}(\mathbb{R}^n)\| \tag{A.31}$$

for all

$$f \in F^s_{pq}(\mathbb{R}^n) \quad \textit{with } \operatorname{supp} f \subset \{x : |x| < \varepsilon\}, \tag{A.32}$$

where the equivalence constants in (A.31) *are independent of* ε *and* f *with* (A.32).

(ii) *Then there is a constant c such that*

$$\|\varphi f \,|F^s_{pq}(\mathbb{R}^n)\| \leqslant c\, \|f\,|F^s_{pq}(\mathbb{R}^n)\|$$

for all $0 < \varepsilon \leqslant 1$, *all* f *according to* (A.32) *and all* φ *having classical derivatives up to order* $1 + [s]$ *with*

$$|D^\gamma \varphi(x)| \leqslant \varepsilon^{-|\gamma|}, \quad |\gamma| \leqslant 1 + [s], \quad |x| < 2\varepsilon.$$

Remark 43. These assertions are covered by [17, Sections 5.16, 5.17].

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